9

To Mock a Mockingbird

A certain enchanted forest is inhabited by talking birds. Given any birds A and B, if you call out the name of B to A, then A will respond by calling out the name of some bird to you; this bird we designate by AB. Thus AB is the bird named by A upon hearing the name of B. Instead of constantly using the cumbersome phrase "A's response to hearing the name of B," we shall more simply say: "A's response to B." Thus AB is A's response to B. In general, A's response to B is not necessarily the same as B's response to A—in symbols, AB is not necessarily the same bird as BA. Also, given three birds A, B, and C, the bird A(BC) is not necessarily the same as the bird (AB)C. The bird A(BC) is A's response to the bird BC, whereas the bird (AB)C is the response of the bird AB to the bird C. The use of parentheses is thus necessary to avoid ambiguity; if I just wrote ABC, you could not possibly know whether I meant the bird A(BC) or the bird (AB)C.

Mockingbirds: By a *mockingbird* is meant a bird M such that for any bird x, the following condition holds:

$$Mx = xx$$

M is called a mockingbird for the simple reason that its response to any bird x is the same as x's response to itself—in other words, M *mimics* x as far as its response to x goes. This means that if you call out x to M or if you call out x to itself, you will get the same response in either case.*

Composition: The last technical detail before the fun starts

* For handy reference to the birds, each is alphabetically listed in "Who's Who Among the Birds," p. 244.

is this: Given any birds A, B, and C (not necessarily distinct) the bird C is said to *compose* A with B if for every bird x the following condition holds:

$$Cx = A(Bx)$$

In words, this means that C's response to x is the same as A's response to B's response to x.

TO MOCK A MOCKINGBIRD

1 • The Significance of the Mockingbird

It *could* happen that if you call out B to A, A might call the same bird B back to you. If this happens, it indicates that A is *fond* of the bird B. In symbols, A is fond of B means that AB = B.

We are now given that the forest satisfies the following two conditions.

C_1 (the composition condition): For any two birds A and B (whether the same or different) there is a bird C such that for any bird x, Cx = A(Bx). In other words, for any birds A and B there is a bird C that composes A with B.

C_2 (the mockingbird condition): The forest contains a mockingbird M.

One rumor has it that every bird of the forest is fond of at least one bird. Another rumor has it that there is at least one bird that is not fond of any bird. The interesting thing is that it is possible to settle the matter completely by virtue of the given conditions C_1 and C_2.

Which of the two rumors is correct?

Note: This is a basic problem in the field known as *combinatory logic.* The solution, though not lengthy, is extremely ingenious. It is based on a principle that derives ultimately from the work of the logician Kurt Gödel. This principle will permeate parts of many of the chapters that follow.

2 • Egocentric?

A bird x is called *egocentric* (sometimes *narcissistic*) if it is fond of itself—that is, if x's response to x is x. In symbols, x is egocentric if xx = x.

The problem is to prove that under the given conditions C_1 and C_2 of the last problem, at least one bird is egocentric.

3 • Story of the Agreeable Bird

Two birds A and B are said to *agree* on a bird x if their responses to x are the same—in other words if Ax = Bx. A bird A is called *agreeable* if for every bird B, there is at least one bird x on which A and B agree. In other words, A is *agreeable* if for every bird B there is a bird x such that Ax = Bx.

We now consider the following variant of Problem 1: We are given the composition condition C_1, but we are not given that there is a mockingbird; instead, we are given that there is an agreeable bird A. Is this enough to guarantee that every bird is fond of at least one bird?

A bonus question: Why is Problem 1 nothing more than a special case of Problem 3? *Hint:* Is a mockingbird necessarily agreeable?

4 • A Question on Agreeable Birds

Suppose that the composition condition C_1 of Problem 1 holds and that A, B, and C are birds such that C composes A with B. Prove that if C is agreeable then A is also agreeable.

5 • An Exercise in Composition

Again suppose that condition C_1 holds. Prove that for any birds A, B, and C there is a bird D such that for every bird x, Dx = A(B(Cx)). This fact is quite useful.

6 • Compatible Birds

Two birds A and B, either the same or different, are called *compatible* if there is a bird x and a bird y, either the same or different, such that Ax = y and By = x. This means that if you call out x to A then you will get y as a response, whereas if you call out y to B, you will get x as a response.

Prove that if conditions C_1 and C_2 of Problem 1 hold, then any two birds A and B are compatible.

7 • Happy Birds

A bird A is called *happy* if it is compatible with itself. This means that there are birds x and y such that Ax = y and Ay = x.

Prove that any bird that is fond of at least one bird must be a happy bird.

8 • Normal Birds

We will henceforth call a bird *normal* if it is fond of at least one bird. We have just proved that every normal bird is happy. The converse is not necessarily true; a happy bird is not necessarily normal.

Prove that if the composition condition C_1 holds and if there is at least one happy bird in the forest, then there is at least one normal bird.

HOPELESS EGOCENTRICITY

9 • Hopelessly Egocentric

We recall that a bird B is called *egocentric* if BB = B. We call a bird B *hopelessly egocentric* if for *every* bird x, Bx = B. This means that whatever bird x you call out to B is irrelevant; it only calls B back to you! Imagine that the bird's name is Bertrand. When you call out "Arthur," you get the response "Bertrand"; when you call out "Raymond," you get the response "Bertrand"; when you call out "Ann," you get the response "Bertrand." All this bird can ever think of is itself!

More generally, we say that a bird A is *fixated* on a bird B if for every bird x, Ax = B. That is, all A can think of is B! Then a bird is hopelessly egocentric just in the case that it is fixated on itself.

A bird K is called a *kestrel* if for any birds x and y, (Kx)y = x. Thus if K is a kestrel, then for every bird x, the bird Kx is fixated on x.

Given conditions C_1 and C_2 of Problem 1, and the existence of a kestrel K, prove that at least one bird is hopelessly egocentric.

10 • Fixation

If x is fixated on y, does it necessarily follow that x is fond of y?

11 • A Fact About Kestrels

Prove that if a kestrel is egocentric, then it must be hopelessly egocentric.

12 • Another Fact About Kestrels

Prove that for any kestrel K and any bird x, if Kx is egocentric then K must be fond of x.

13 • A Simple Exercise

Determine whether the following statement is true or false: If a bird A is hopelessly egocentric, then for any birds x and y, Ax = Ay.

14 • Another Exercise

If A is hopelessly egocentric, does it follow that for any birds x and y, (Ax)y = A?

15 • Hopeless Egocentricity Is Contagious!

Prove that if A is hopelessly egocentric, then for every bird x, the bird Ax is also hopelessly egocentric.

16 • Another Fact About Kestrels

In general, it is not true that if Ax = Ay then x = y. However, it *is* true if A happens to be a kestrel K. Prove that if Kx = Ky then x = y. (We shall henceforth refer to this fact as the *left cancellation law for kestrels.*)

17 • A Fact About Fixation

It is possible that a bird can be fond of more than one bird, but it is not possible for a bird to be fixated on more than one bird. Prove that it is impossible for a bird to be fixated on more than one bird.

18 • Another Fact About Kestrels

Prove that for any kestrel K and any bird x, if K is fond of Kx, then K is fond of x.

19 • A Riddle

Someone once said: "Any egocentric kestrel must be extremely lonely!" Why is this true?

IDENTITY BIRDS

A bird I is called an *identity* bird if for every bird x the following condition holds:

$$Ix = x$$

The identity bird has sometimes been maligned, owing to the fact that whatever bird x you call to I, all I does is to echo x back to you. Superficially, the bird I appears to have no intelligence or imagination; all it can do is repeat what it hears. For this reason, in the past, thoughtless students of ornithology referred to it as the *idiot* bird. However, a more profound ornithologist once studied the situation in great depth and discovered that the identity bird is in fact highly intelligent! The *real* reason for its apparently unimaginative behavior is that it has an unusually large heart *and hence is fond of every bird!* So when you call x to I, the reason it responds by calling back x is not that it can't think of anything else; it's just that it wants you to know that it is fond of x!

Since an identity bird is fond of every bird, then it is also fond of itself, so every identity bird is egocentric. However, its egocentricity doesn't mean that it is any more fond of itself than of any other bird!

Now for a few simple problems about identity birds.

• 20 •

Supposing we are told that the forest contains an identity bird I and that I is agreeable, in the sense of Problem 3. Does it follow that every bird must be fond of at least one bird? *Note:* We are no longer given conditions C_1 and C_2.

• 21 •

Suppose we are told that there is an identity bird I and that every bird is fond of at least one bird. Does it necessarily follow that I is agreeable?

• 22 •

Suppose we are told that there is an identity bird I, but we are not told whether I is agreeable or not. However, we are told that every pair of birds is compatible, in the sense of Problem 6. Which of the following conclusions can be validly drawn?

1. Every bird is normal—i.e., fond of at least one bird.
2. I is agreeable.

23 • Why?

The identity bird I, though egocentric, is in general not *hopelessly* egocentric. Indeed, if there were a hopelessly egocentric identity bird, the situation would be quite sad. Why?

LARKS

A bird L is called a *lark* if for any birds x and y the following holds:

$$(Lx)y = x(yy)$$

Larks have some interesting properties, as we will now see.

• 24 •

Prove that if the forest contains a lark L and an identity bird I, then it must also contain a mockingbird M.

• 25 •

One reason I like larks is this: If there is a lark in the forest, then it follows without further ado that every bird is fond of at least one bird. And so you see, the lark has a wonderful effect on the forest as a whole; its presence makes every bird normal. And since all normal birds are happy, by Problem 7, then a lark L in the forest causes all the birds to be happy!

Why is this true?

26 • Another Riddle

Why is a hopelessly egocentric lark unusually attractive?

• 27 •

Assuming that no bird can be both a lark and a kestrel—as any ornithologist knows!—prove that it is impossible for a lark to be fond of a kestrel.

• 28 •

It might happen, however, that a kestrel K is fond of a lark L. Show that if this happens, then every bird is fond of L.

• 29 •

Now let me tell you the most surprising thing I know about larks: Suppose we are given that the forest contains a lark L and we are not given any other information. From just this one fact alone, it can be proved that at least one bird in the forest must be egocentric!

The proof of this is a bit tricky. Given the lark L, we can actually write down an expression for an egocentric bird—and we can write it using just the letter L, with parentheses, of course. The shortest expression that I have been able to find has a length of 12, not counting parentheses. That is, we can write L twelve times and then by parenthesizing it the right way, have the answer. Care to try it? Can you find a shorter expression than mine that works? Can it be proved that there is no shorter expression in L that works? I don't know! At any rate, see if you can find an egocentric bird, given the bird L.

SOLUTIONS

1 · The first rumor is correct; every bird A is fond of at least one bird. We prove this as follows:

Take any bird A. Then by condition C_1, there is a bird C that composes A with the mockingbird M, because for *any* bird B, there is a bird C that composes A with B, so this is also true if B happens to be the mockingbird M. Thus for any bird x, Cx = A(Mx), or what is the same thing, A(Mx) = Cx. Since this equation holds for *every* bird x, then we can substitute C for x, thus getting the equation A(MC) = CC.

But MC = CC, since M is a mockingbird, and so in the equation A(MC) = CC, we can substitute CC for MC, thus getting the equation A(CC) = CC. This means that A is fond of the bird CC!

In short, if C is any bird that composes A with M, then A is fond of the bird CC. Also, A is fond of MC, since MC is the same as the bird CC.

2 • We have just seen that conditions C_1 and C_2 imply that every bird is fond of at least one bird. This means, in particular, that the mockingbird M is fond of at least one bird E. Now we show that E must be egocentric.

First, ME = E, since M is fond of E. But also ME = EE, because M is a mockingbird. So E and EE are both identical with the bird ME, so EE = E. This means that E is fond of E—i.e., that E is egocentric.

Remark: Since E is egocentric and E = ME, then ME is egocentric. Doesn't the word "ME" tell its own tale?

3 • We are given that the composition condition C_1 holds and that there is an agreeable bird A.

Take any bird x. By the composition condition, there is some bird H that composes x with A. Since A is agreeable, then A agrees with H on some bird y. We will show that x must be fond of the bird Ay.

Since A agrees with H on y, then Ay = Hy. But since H composes x with A, then Hy = x(Ay). Therefore Ay = Hy = x(Ay), and so Ay = x(Ay), or what is the same thing, x(Ay) = Ay. This means that x is fond of Ay.

A bonus question: The mockingbird is certainly agreeable, because for any bird x, M agrees with x on the very bird x, since Mx = xx. In other words there *is* a bird y—namely x itself—such that My = xy.

Since every mockingbird is agreeable, then the given conditions of Problem 3 imply the given conditions of Problem 1, and therefore the solution of Problem 1 gives an alternative solution to Problem 3, though a more complicated one.

4 • We are given that C composes A with B and that C is agreeable. We are also given the composition condition. We are to show that A is agreeable.

Take any bird D. We must show that A agrees with D on some bird or other. Since the composition law holds, then there is a bird E that composes D with B. Also C agrees with E on some bird x, because C is agreeable—thus Cx = Ex. Also Ex = D(Bx), because E composes D with B, and Cx = A(Bx), because C composes A with B. Therefore, since Ex = D(Bx), we have A(Bx) = D(Bx). And so A agrees with D on the bird Bx. This proves that for *any* bird D, there is a bird on which A and D agree, which means that A is agreeable.

In short, A(Bx) = Cx = Ex = D(Bx).

5 • Suppose the composition law C_1 holds. Take any birds A, B, and C. Then there is a bird E that composes B with C, and so for any bird x, Ex = B(Cx), and hence A(Ex) = A(B(Cx)). Using the composition law again, there is a bird D that composes A with E, and hence Dx = A(Ex). Therefore Dx = A(Ex) = A(B(Cx)), and so Dx = A(B(Cx)).

6 • We are given that conditions C_1 and C_2 of Problem 1 hold. Therefore every bird is fond of at least one bird, according to the solution to Problem 1. Now take any birds A and B. By condition C_1, there is a bird C that composes A with B. The bird C is fond of some bird—call it y. Thus Cy = y. Also Cy = A(By)—because C composes A with B. Therefore A(By) = y. Let x be the bird By. Then Ax = y, and of course By = x. This proves that A and B are compatible.

7 • To say that A is compatible with B doesn't necessarily mean that there are two *distinct* birds x and y such that Ax = y and By = x; x and y may be the same bird. So if there is a bird x such that Ax = x and Bx = x, that surely implies that A and B are compatible. Thus if Ax = x, then A is automatically compatible with A, because Ax = y and Ay = x, when y is the same bird as x.

Therefore if A is fond of x, then Ax = x, and A is compatible with A, which means that A is happy.

8 • Suppose H is a happy bird. Then there are birds x and y such that Hx = y and Hy = x. Since Hx = y, then we can substitute Hy for x (since Hy = x) and obtain H(Hy) = y. Also, by the composition condition C_1, there is a bird B that composes H with H, and so By = H(Hy) = y. So By = y, which means that B is fond of y. Since B is fond of some bird y, then B is normal.

9 • We are given the conditions of Problem 1, hence every bird is fond of at least one bird. In particular, the kestrel K is fond of some bird A. Thus KA = A. Hence for every bird x, (KA)x = Ax. Also (KA)x = A, since K is a kestrel. Therefore Ax = A. Since for every bird x, Ax = A, then A is hopelessly egocentric.

We can also look at the matter this way: If the kestrel K is fond of a bird A, then KA = A. Also KA is fixated on A, and since KA = A, then A is fixated on A, which means that A is hopelessly egocentric. And so we see that any bird of which the kestrel is fond must be hopelessly egocentric.

10 • Of course it does! If x is fixated on y, then for *every* bird z, xz = y, hence in particular, xy = y, which means that x is fond of y.

11 • If K is egocentric, then K is fond of K. But we proved in Problem 9 that any bird of which K is fond must be hope-

lessly egocentric, and so if K is egocentric, K is hopelessly egocentric.

12 · Suppose that Kx is egocentric. Then (Kx)(Kx) = Kx. But also (Kx)(Kx) = x, because for *any* bird y, (Kx)y = x, so this is also true when y is the bird Kx. Therefore Kx = x, because Kx and x are both equal to the bird (Kx)(Kx), so they are equal to each other. This means that K is fond of x.

13 · Suppose A is hopelessly egocentric. Then Ax = A and Ay = A, so Ax = Ay; they are both equal to A. Thus the statement is true.

14 · Yes, it does follow. Suppose A is hopelessly egocentric. Then Ax = A, hence (Ax)y = Ay and Ay = A, so (Ax)y = A.

15 · Suppose A is hopelessly egocentric. Then for any birds x and y, (Ax)y = A, according to the last problem. Also Ax = A, since A is hopelessly egocentric. Therefore (Ax)y = Ax, since (Ax)y and Ax are both equal to A. Therefore, for any bird y, (Ax)y = Ax, which means that Ax is hopelessly egocentric.

16 · Suppose Kx = Ky and that K is a kestrel. Then for any bird z, (Kx)z = (Ky)z. But (Kx)z = x and (Ky)z = y, so x = (Kx)z = (Ky)z = y. Therefore x = y.

17 · Suppose A is fixated on x and A is fixated on y; we will show that x = y.

Take any bird z. Then Az = x, since A is fixated on x, and Az = y, since A is fixated on y. Therefore x and y are both equal to the bird Az, and so x = y.

18 · Suppose K is fond of Kx. Then K(Kx) = Kx. Now, K(Kx) is fixated on Kx, whereas Kx is fixated on x. But since

K(Kx) and Kx are the same bird, then the same bird is fixated on both Kx and x, which makes Kx = x, according to the last problem. Therefore K is fond of x.

19 • We will show that the only way a kestrel can be egocentric is that it is the only bird in the forest!

Proof 1: Suppose that K is an egocentric kestrel. Then K is hopelessly egocentric, according to Problem 11. Now let x and y be any birds in the forest, and we will show that x = y.

Since K is hopelessly egocentric, then Kx = K and Ky = K, so Kx = Ky. Therefore, according to Problem 16, x = y. So any birds x and y in the forest are identical with each other, and there is only one bird in the forest. Since we are given that K is in the forest, then K is the only bird in the forest.

Proof 2: Again we use the fact that since K is egocentric, then K is hopelessly egocentric. Now let x be any bird in the forest. Then Kx is fixated on x, since K is a kestrel, and also Kx = K, since K is hopelessly egocentric. Therefore K is fixated on x, since Kx is fixated on x and Kx is the bird K. This proves that K is fixated on every bird x in the forest. But by Problem 17, K cannot be fixated on more than one bird, hence all the birds of the forest must be identical.

20 • Yes, it does. Suppose I is agreeable. Then for any bird x there is a bird y such that xy = Iy. But Iy = y, hence xy = y. Thus x is fond of y.

21 • Yes, it does. Suppose every bird x is fond of some bird y. Then xy = y, but also Iy = y, and so bird I agrees with x on the bird y.

22 • Both conclusions follow.

1. We are given that I is an identity bird and that any two birds are compatible. Now, take any bird B. Then B is compatible with I, so there are birds x and y such that Bx = y

and Iy = x. Since Iy = x, then y = x, because y = Iy. Since y = x and Bx = y, then Bx = x, so B is fond of the bird x. Therefore every bird B is fond of some bird x.

2. This follows from the first conclusion and Problem 21.

23 · Suppose I is an identity bird and I is hopelessly egocentric. Take any bird x. Then Ix = I, since I is hopelessly egocentric, but also Ix = x, since I is an identity bird. Then x = I, so again we have the sad fact that there is only one bird in the forest; every bird x is identical with I.

24 · Suppose L is a lark and I is an identity bird. Then for any bird x, (LI)x = I(xx) = xx. Therefore LI is a mockingbird. This means that if someone calls out I to L, then L names a mockingbird.

25 · This is quite simple. Suppose L is a lark. Then for any birds x and y, (Lx)y = x(yy). This is also true when y is the bird Lx, and so (Lx)(Lx) = x((Lx)(Lx)). And so, of course, x((Lx)(Lx)) = (Lx)(Lx), which means that x is fond of the bird (Lx)(Lx). So every bird x is normal.

For help in solving future problems, we make a note of the fact that for any lark L, any bird x is fond of the bird (Lx)(Lx).

26 · We will show that if L is a hopelessly egocentric lark, then every bird is fond of L.

Suppose L is a lark and that L is hopelessly egocentric. Since L is hopelessly egocentric, then for any birds x and y, (Lx)y = L, according to Problem 14. In particular, taking Lx for y, (Lx)(Lx) = L. But x is fond of (Lx)(Lx), as we proved in the last problem. Therefore x is fond of L, since (Lx)(Lx) = L. This proves that every bird x is fond of L.

27 · This is an interesting proof! We have already proved in Problem 18 that if K is fond of Kx, then K is fond of x. In particular, taking K for x, if K is fond of KK, then K is fond of K.

Now suppose L is a lark of the forest and K is a kestrel of the forest and that L is fond of K. Then LK = K, hence (LK)K = KK. But (LK)K = K(KK), since L is a lark. Therefore KK = K(KK)—they are both equal to (LK)K—which makes K fond of KK. Then K is fond of K, as we showed in the last paragraph. Hence K is egocentric. Then by Problem 19, K is the only bird in the forest. But this contradicts the given fact that L is in the forest and L ≠ K.

28 • Suppose K is fond of L. Then by the solution to Problem 9, L is hopelessly egocentric. Therefore, by Problem 26, every bird is fond of L.

29 • Suppose the forest contains a lark L. Then by Problem 25, every bird is fond of at least one bird. In particular, the bird LL is fond of some bird y. (This constitutes our first trick!) Therefore (LL)y = y, but (LL)y = L(yy), because L is a lark, and so for *any* bird x, (Lx)y = x(yy). Therefore L(yy) = y, since they are both equal to (LL)y. Therefore (L(yy))y = yy. (This is our second trick!) But (L(yy))y = (yy)(yy). This can be seen by substituting (yy) for x in the equation (Lx)y = x(yy). So yy and (yy)(yy) are both equal to (L(yy))y, hence (yy)(yy) = yy, which means that yy is egocentric.

This proves that if y is any bird of whom LL is fond, then yy must be egocentric. Furthermore, LL *is* fond of some bird y, according to Problem 25.

We can actually compute a bird y of which LL is fond. We saw in the solution to Problem 25 that for any bird x, x is fond of (Lx)(Lx). Therefore LL is fond of (L(LL))(L(LL)). So we can take (L(LL))(L(LL)) for the bird y. Our egocentric bird is then ((L(LL))(L(LL)))((L(LL))(L(LL))).

◐ 10 ◑

Is There a Sage Bird?

Inspector Craig of Scotland Yard was a man of many interests. His activities in crime detection, law, logic, number machines, retrograde analysis, vampirism, philosophy, and theology are familiar to readers of my earlier puzzle books. He was equally interested in ornithological logic—a field that applies combinatory logic to the study of birds. He was therefore delighted to hear about the bird forest of the last chapter and decided to visit it and do some "inspecting."

When he arrived, the first thing he did was to interview the bird sociologist of the forest, whose name was Professor Fowler. Professor Fowler told Craig of the two laws C_1 and C_2, the basic composition law and the existence of a mockingbird, from the first problem of the last chapter. From this, Inspector Craig was of course able to deduce that every bird was fond of at least one bird.

"However," explained Craig to Fowler, "I would like to go a bit more deeply into the matter. I am what mathematical logicians call a constructivist. I am not satisfied to know merely that given any bird x, there exists *somewhere* in the forest a bird y of which x is fond; I would like to know how, given a bird x, I can *find* such a bird y. Is there by any chance a bird in this forest that can supply such information?"

"I really don't understand your question," replied Fowler. "What do you mean by a bird's *supplying* such information?"

"What I want to know," said Craig, "is whether or not there is some special bird which, whenever I call out the name of a bird x to it, will respond by naming a bird of which x is fond. Do you know whether there is such a bird?"

"Oh, now I understand what you mean," said Fowler, "and your question is a very interesting one! All I can tell you is that it has been *rumored* that there is such a bird, but its existence in this forest has not been substantiated. Such birds are called *sage birds*—sometimes *oracle birds*—but, as I said, we don't know if there are any sage birds here. According to some history books, whose authenticity, however, is uncertain, sage birds were first observed in Greece—in Delphi, in fact—which might account for their also being called oracle birds. Accordingly, the Greek letter Θ is used to denote a sage bird. If there really is such a bird, then it has the remarkable property that for any bird x, x is fond of the bird Θx—in other words, $x(\Theta x) = \Theta x$. Or, as you might put it, if you call out x to Θ, then Θ will name a bird of which x is fond.

"I have been trying to find a sage bird for a long time now, but I'm afraid I haven't been very successful. If you could throw any light on the matter, I would be enormously grateful!"

Inspector Craig rose, thanked Professor Fowler, and told him that he would devote some thought to the matter. Craig then spent the day walking through the forest concentrating deeply on the problem. The next morning he returned to Professor Fowler.

"I doubt very much," said Craig, "that—from just the two conditions C_1 and C_2 that you have told me—it can be determined whether or not this forest contains a sage bird.

"The trouble is this," he explained: "We know that there is a mockingbird M. And we know that for any bird x there is *some* bird y that composes x with the mockingbird M. Then, as you know, x is fond of the bird yy. But given the bird x, how does one *find* a bird y that composes x with M? If there were some bird A that supplied this information, then the problem would be solvable. But from what you have told me, I have no reason to believe that there is such a bird."

"Oh, but there *is* such a bird," replied Fowler. "I'm sorry, but I forgot to tell you that we do have a bird A such that whatever bird x you call out to A, A will respond by naming a bird that composes x with M. That is, for any bird x, the bird Ax composes x with M."

"Splendid!" said Craig. "That completely solves your problem: This forest *does* contain a sage bird."

How did Craig know this?

"Wonderful!" said Fowler, after Craig proved that the forest contained a sage bird.

"And now, what are your plans? You know, perhaps, that this forest is only one of a whole chain of remarkable bird forests. You should definitely visit Curry's Forest, and before you come to that, you will pass through a forest unusually rich in bird life. You will probably want to spend a good deal of time there; there is so much to learn!"

Craig thanked Professor Fowler and departed for the next forest. He little realized that this was only the beginning of a summer-long venture!

SOLUTION

This problem, though important, is really quite simple!

To begin with, the bird A described by Fowler is nothing more nor less than a lark! The reason is this: To say that for every bird x, the bird Ax composes x with M is to say that for any bird x and any bird y, (Ax)y = x(My). But My = yy, so x(My) = x(yy). Therefore the bird A described by Fowler satisfies the condition that for any birds x and y, (Ax)y = x(yy), which means that A is a lark.

And so the problem boils down to this: Given a mockingbird M, a lark L, and the basic composition condition C_1, prove that the forest contains a sage bird.

Well, we have shown in the solution to Problem 25 of the last chapter that any bird x is fond of the bird (Lx)(Lx), hence x is fond of M(Lx), since M(Lx) = (Lx)(Lx). Now, by the basic composition condition C_1, there is a bird Θ that composes M with L. This means that for any bird x, Θx = M(Lx). Since x is fond of M(Lx) and M(Lx) = Θx, then x is fond of Θx, which means that Θ is a sage bird.

In short, any bird that composes M with L is a sage bird.

The theory of sage birds (technically called *fixed point combinators*) is a fascinating and basic part of combinatory logic; we have only scratched the surface. We will go more deeply into the theory of sage birds in a later chapter, but we must first turn our attention to some of the more basic birds, which we will do in the next two chapters.

◐ 11 ◑
Birds Galore

In the next bird forest Craig visited, the resident bird sociologist was named Professor Adriano Bravura. Professor Bravura had an aristocratic, somewhat proud bearing, which many mistook for haughtiness. Craig soon realized that this impression was quite misleading; Professor Bravura was an extremely dedicated scholar who, like many scholars, was often absentminded and abstracted, and this "abstractedness" was what was so often mistaken for detachment and lack of concern for other human beings. Actually, Professor Bravura was a very warmhearted person who took a great interest in his students. Craig learned an enormous amount from him—as will the reader!

"We have many, many interesting birds in this forest," said Bravura to Craig at the first interview, "but before I tell you about them, it will be best for me to explain to you a well-known abbreviation concerning parentheses."

Professor Bravura then took a pencil and a pad of paper and placed it so that Craig could see what he was writing.

"Suppose I write down xyz," said Bravura. "Without further explanation, this notation is ambiguous; you cannot know whether I mean (xy)z or x(yz). Well, the convention is that we will mean (xy)z—or, as we say, if parentheses are omitted, they are to be restored *to the left.* This is the tradition in combinatory logic, and after a little practice, it makes complex expressions more easily readable.

"The same convention applies to even more complex expressions—for example, let us look at the expression (xy)zw. We look at (xy) as a unit, and so (xy)zw is really ((xy)z)w. What is xyzw? We first restore parentheses to the leftmost part, which is xy, and so xyzw is (xy)zw, which in turn is ((xy)z)w. And so xyzw is simply an abbreviation for ((xy)z)w.

"Other examples," said Bravura: "x(yz)w = (x(yz))w, whereas x(yzw) = x((yz)w).

"I think you should now try the following exercises to be sure that you fully grasp the principle of restoring parentheses to the left."

Here are the exercises Bravura gave Craig; the answers are given immediately afterward.

Exercises: In each of the following cases, fully restore parentheses to the left.

a. xy(zwy)v = ?

b. (xyz)(wvx) = ?

c. xy(zwv)(xz) = ?

d. xy(zwv)xz = ? *Note:* The answer is different from that for (c)!

e. x(y(zwv))xz = ?

f. Is the following true or false?

$$xyz(AB) = (xyz)(AB)$$

g. Suppose $A_1 = A_2$. Can we conclude that $BA_1 = BA_2$? And can we conclude that $A_1B = A_2B$?

h. Suppose xy = z. Which of the following conclusions is valid? *Note:* Tricky and important!

1. xyw = zw
2. wxy = wz

Answers:

a. xy(zwy)v = ((xy)((zw)y))v

b. (xyz)(wvx) = ((xy)z)((wv)x)

c. xy(zwv)(xz) = ((xy)((zw)v))(xz)

d. xy(zwv)xz = (((xy)((zw)v))x)z

e. x(y(zwv))xz = ((x(y((zw)v)))x)z
f. True; both sides reduce to ((xy)z)(AB).
g. Both conclusions are correct.
h. Suppose xy = z.

1. xyw = zw says that (xy)w = zw, and this is correct, since the birds (xy) and xy are identical and we are given that xy = z, and hence (xy) = z.

2. wxy = wz says that (wx)y = wz, and this certainly does *not* follow from the fact that xy = z! What *does* follow is that w(xy) = wz, but this is very different from (wx)y = wz.

So the first conclusion follows, but the second does not.

BLUEBIRDS

"Now that we have gone through these preliminaries," said Bravura, "we can get on to the more interesting things about this forest.

"As I have told you, we have many fascinating birds here. A bird of basic importance is the *bluebird*—by which I mean a bird B such that for all birds x, y, z, the following holds:

Bxyz = x(yz)

"In unabbreviated notation," said Bravura, "I would have written: ((Bx)y)z = x(yz). However, I find it much easier to read: Bxyz = x(yz)."

• 1 •

"Why are bluebirds of basic importance?" asked Craig.

"For many reasons, which you will see," replied Bravura. "For one thing, if a forest contains a bluebird—which this forest fortunately does—then the basic composition law must hold: For any bird C and D, there is a bird E that composes C with D. Can you see why?"

Note: Recall from Chapter 8 that if E composes C with D, it means that for every bird x, Ex = C(Dx).

2 • Bluebirds and Mockingbirds

"Suppose," said Bravura, "that a bird forest contains a bluebird B and a mockingbird M. Since B is present, the composition law holds, as you have just seen. Therefore, as you know, it follows that every bird x is fond of some bird. However, since B is present, you can write down an expression in terms of B, M, x that describes a bird of which x is fond. Can you see how to write down such an expression?"

3 • Egocentric

"Given a bluebird B and a mockingbird M," said Bravura, "can you see how to write down an expression for an egocentric bird?"

4 • Hopelessly Egocentric

"Now," said Bravura, "suppose a forest contains a bluebird B, a mockingbird M, and a kestrel K. See if you can write down an expression in terms of B, M, and K for a hopelessly egocentric bird."

SOME DERIVATIVES OF THE BLUEBIRD

"And now," said Bravura, "let us forget about mockingbirds and kestrels for a while and concentrate on just the bluebird B. From just this one bird alone, many useful birds can be derived. Not all of them are of major importance, but several of them will crop up from time to time in the course of your study."

5 • Doves

"For example, one fairly important bird is the *dove*, by which is meant a bird D such that for any birds x, y, z, w, the following condition holds:

$$Dxyzw = xy(zw)$$

"The bird D can be derived from B alone. Can you see how?"

6 • Blackbirds

"Then," said Bravura, "there is the *blackbird*—a bird B_1 such that for any birds x, y, z, w, the following condition holds:

$$B_1xyzw = x(yzw)$$

"Prove that any forest containing a bluebird must also contain a blackbird.

"Of course," added Bravura, "in deriving a blackbird from a bluebird, you are free to use the dove D if that is helpful, since you have already seen how D can be derived from B."

7 • Eagles

"Then there is the *eagle*," said Bravura, "by which is meant a bird E such that for any birds x, y, z, w, v, the following condition holds:

$$Exyzwv = xy(zwv)$$

"The eagle can be derived from just the bird B. Can you see how? Again, it will simplify your derivation to use birds already derived from B."

8 • Buntings

"A *bunting*," said Bravura, "is a bird B_2 satisfying the following condition—for any birds x, y, z, w, v, of course:

$$B_2xyzwv = x(yzwv)$$

"Given B, find a bunting B_2."

9 • Dickcissels

Bravura continued: "By a *dickcissel* I mean a bird D_1 satisfying the following condition:

$$D_1xyzwv = xyz(wv)$$

"Show how a dickcissel D_1 can be derived from a bluebird B."

10 • Becards

"Then there is the *becard*," said Bravura, "a bird B_3 such that for all birds x, y, z, w, the following condition holds:

$$B_3xyzw = x(y(zw))$$

"Can you see how to derive a becard from a bluebird, and from any other birds already derived from B?"

11 • Dovekies

"Then there is the *dovekie*," said Bravura, "which is a bird D_2 satisfying the following condition:

$$D_2xyzwv = x(yz)(wv)$$

"Can you see how to derive a dovekie D_2 from a bluebird B?"

12 • Bald Eagles

"And now," said Bravura, "given a bluebird B, see if you can derive a *bald eagle*—a bird Ê such that for all birds x, y_1, y_2, y_3, z_1, z_2, z_3, the following condition holds:

$$\hat{E}xy_1y_2y_3z_1z_2z_3 = x(y_1y_2y_3)(z_1z_2z_3)."$$

"I think you have had enough problems for today," said Bravura. "We have now derived eight different birds from the one bird B. We could derive many more, but I think you have seen enough to get a good feeling for the behavior of the bluebird. All these birds—including B—belong to a family of birds known as *compositors*. They serve to introduce parentheses. The only two that you need remember are the bluebird B and the dove D; they are standard in the literature of combinatory logic. The other seven birds don't have standard names, but I have found it convenient to give them names, as some of them will crop up again.

"Tomorrow, I will tell you about some very different birds."

SOME OTHER BIRDS

Inspector Craig returned bright and early the next morning. He was surprised to find Professor Bravura in the garden, seated at a table with paper, pencils, and piles of notes. Two cups of freshly brewed steaming hot coffee had been laid out.

13 • Warblers

"Too beautiful a morning to work indoors," said Bravura. "Besides, I may be able to show you some of the birds we discuss.

"Ah, there goes a warbler!" said Bravura. "This bird W is an important bird and is quite standard in combinatory logic. It is defined by the following condition:

$$Wxy = xyy$$

"Do not confuse this with the lark L!" cautioned Bravura. "Remember, $Lxy = x(yy)$, whereas $Wxy = xyy$. These are very different birds!

"I have a nice little problem for you," continued Bravura. "Prove that any forest containing a warbler W and a kestrel K must contain a mockingbird M."

After a bit of time, Bravura said, "I see you are having difficulty. I think I will first give you two simpler problems."

• 14 •

"Show that from a warbler W and an identity bird I we can get a mockingbird."

Craig solved this quite easily.

• 15 •

"Now show that from a warbler W and a kestrel K we can get an identity bird."

"Oh, I get the idea!" said Craig.

16 • Cardinals

Just then, a brilliant red bird flew by.

"A cardinal!" said Bravura. "One of my favorite birds! It also plays a basic role in combinatory logic. The cardinal C is defined by the following condition:

$$Cxyz = xzy$$

"The cardinal belongs to an important family of birds known as *permuting* birds. You see that in the above equation, the variables y and z have got switched around.

"Here's an easy problem for you," said Bravura. "Prove that any forest containing a cardinal and a kestrel must contain an identity bird."

17 • Thrushes

"A bird closely related to the cardinal is the thrush," said Bravura. "Why, there is one right over there! A thrush T is defined by the following condition:

$$Txy = yx$$

"The thrush is the simplest of the permuting birds," said Bravura. "It is derivable from a cardinal C and an identity bird I. Can you see how?"

18 • Commuting Birds

"Two birds x and y are said to commute," said Bravura, "if $xy = yx$. This means that it makes no difference whether you call out y to x, or x to y; you get the same response in either case.

"There's an interesting thing about thrushes," said Bravura. "If a forest contains a thrush, and if every bird of the forest is fond of some bird, then there must be at least one bird A that commutes with every bird. Can you see how to prove this?"

• 19 •

"Given a bluebird B, a thrush T, and a mockingbird M," said Bravura, "find a bird that commutes with every bird."

BLUEBIRDS AND THRUSHES

"Bluebirds and thrushes work beautifully together!" said Bravura. "From these two birds, you can derive a whole variety of birds known as *permuting* birds. For one thing, from a bluebird B and a thrush T, you can derive a cardinal—this was discovered by the logician Alonzo Church in 1941."

"That sounds interesting," said Craig. "How is it done?"

"The construction is a bit tricky," said Bravura. "Church's expression for a cardinal C in terms of B and T has eight letters, and I doubt that it can be done with fewer. I will simplify the problem for you by first deriving another bird—one useful in its own right."

20 • Robins

"From B and T," said Bravura, "we can derive a bird R called a *robin* which satisfies the following condition:

$$Rxyz = yzx$$

"Given a bluebird and a thrush, do you see how to derive a robin?"

21 • Robins and Cardinals

"And now, from just the robin alone, we can derive a cardinal. Can you see how? The solution is quite pretty!"

A bonus question: "Putting the last two problems together," said Bravura, "you can see how to derive C from B and T. However, the solution you then get will contain nine letters. It can be shortened by one letter. Can you see how?"

22 • Two Useful Laws

"The following two laws are useful," said Bravura. "We let R be BBT and C be RRR. Prove that for any bird x, the following facts hold:

a. Cx = RxR
b. Cx = B(Tx)R."

23 • A Question

"You have just seen that a cardinal can be derived from a robin. Can a robin be derived from a cardinal?"

24 • Finches

"Ah, there goes a finch!" said Bravura. "A finch is a bird F satisfying the following condition:

Fxyz = zyx

"The finch is another permuting bird, of course, and it can also be derived from B and T. This can be done in several ways. For one thing, a finch can be easily derived from a bluebird, a robin, and a cardinal—and hence from a bluebird and a robin or from a bluebird and a cardinal. Can you see how?"

• 25 •

"Alternatively, a finch can be derived from a thrush T and an eagle E. Can you see how?"

• 26 •

"Now you have available two methods of expressing a finch in terms of a bluebird B and a thrush T. You will see that one of them yields a much shorter expression than the other."

27 • Vireos

"Ah, there goes a vireo!" said Bravura in some excitement. "If you ever get to study combinatorial birds in relation to arithmetic—as doubtless you will—you will find the vireo to be of basic importance. The vireo V is also a permuting bird—it is defined by the following condition:

$$Vxyz = zxy$$

"The vireo has a sort of opposite effect to the robin," commented Bravura. "This bird is also derivable from B and T. One way is to derive it from a cardinal and a finch. Can you see how?"

• 28 •

"How would you most easily express a vireo in terms of a finch and a robin?" asked Bravura. "It can be done with an expression of only three letters."

29 • A Question

"I will later show you another way of deriving a vireo," said Bravura. "Meanwhile I'd like to ask you a question. You have seen that a vireo is derivable from a cardinal and a finch. Is a finch derivable from a cardinal and a vireo?"

30 • A Curiosity

"Another curiosity," said Bravura. "Show that any forest containing a robin and a kestrel must contain an identity bird."

SOME RELATIVES

It was now about noon, and Mrs. Bravura—an exceedingly beautiful, delicate, and refined Venetian lady—brought out a magnificent lunch. After the royal repast, the lesson continued.

"I should now like to tell you about some useful relatives of the cardinal, robin, finch, and vireo," said Bravura. "All of them can be derived from just the two birds B and T—in fact, from B and C."

31 • The Bird C*

"First there is the bird C* called a *cardinal once removed*, satisfying the following condition:

$$C^*xyzw = xywz$$

"Notice," said Bravura, "that in this equation, if we erased x from both sides, and also erased the star, we would have the true statement Cyzw = ywz.

"This is the idea behind the term 'once removed.' The bird C* is like C, except that its action is 'deferred' until we skip over x; we then 'act' on the expression yzw as if we were using a cardinal.

"And now see if you can derive C* from B and C. This is quite simple!"

32 • The Bird R*

"The bird R*—a *robin once removed*—bears much the same relation to R as C* does to C. It is defined by the following condition:

$$R^*xyzw = xzwy$$

"Show that R* is derivable from B and C—and hence from B and T."

33 • The Bird F*

"By a *finch once removed* we mean a bird F* satisfying the following condition:

$$F^*xyzw = xwzy$$

"Now derive F* from birds derivable from B and C."

34 • The Bird V*

"Finally, we have the *vireo once removed*—a bird V* satisfying the following condition:

$$V^*xyzw = xwzy$$

"Show how to derive V* from birds derivable from B and C."

35 • Twice Removed

"Given birds B and C, find birds C**, R**, F**, V** such that for any birds x, y, z_1, z_2, z_3 the following conditions hold:

$$C^{**}xyz_1z_2z_3 = xyz_1z_3z_2$$
$$R^{**}xyz_1z_2z_3 = xyz_2z_3z_1$$
$$F^{**}xyz_1z_2z_3 = xyz_3z_2z_1$$
$$V^{**}xyz_1z_2z_3 = xyz_3z_1z_2$$

"These are the birds C, R, F, V *twice* removed. They will occasionally be useful."

36 • Vireos Revisited

"You have seen that a vireo is derivable from a cardinal and a finch. It is also derivable from the two birds C* and T. Can you see how?"

QUEER BIRDS

"And now," said Bravura, "we turn to an interesting family of birds which both parenthesize and permute. They are all derivable from B and T."

37 • Queer Birds

"The most important member of the family is the *queer* bird Q defined by the following condition:

$$Qxyz = y(xz)$$

"As you can see, Q both introduces parentheses and permutes the order of the letters x and y.

"A comparison of Q with the bluebird B is worth noting: For any birds x and y, the bird Bxy composes x with y, whereas Qxy composes y with x.

"The bird Q is quite easily derived from B and one other bird that you have already derived from B and T. Can you see which one and how?"

38 • Quixotic Birds

"The queer bird Q has several cousins; perhaps the most important one is the *quixotic* bird Q_1 defined by the condition:

$$Q_1xyz = x(zy)$$

"Show that Q_1 is derivable from B and T. Again, you may of course use any birds previously derived from B and T."

39 • Quizzical Birds

"Then there is the *quizzical* bird Q_2—another cousin of Q. It is defined by the condition:

$$Q_2xyz = y(zx)$$

"Show that Q_2 is derivable from B and T."

40 • A Problem

"Here is a little problem for you," said Bravura. "Suppose we are given that a certain bird forest contains a cardinal, but we are not given that it contains a bluebird or a thrush. Prove that if the forest contains either a quixotic bird or a quizzical bird, then it must contain the other as well."

41 • Quirky Birds

"Another cousin of Q is the *quirky* bird Q_3 defined by the following condition:

$$Q_3xyz = z(xy)$$

"Show that Q_3 is derivable from B and T."

42 • Quacky Birds

"The last cousin of Q is the *quacky* bird Q_4 defined by the following condition:

$$Q_4xyz = z(yx)."$$

"What a strange name!" exclaimed Craig.

"I didn't name it; it was named after a certain Professor Quack, who discovered it. Anyhow, can you see how to derive it from B and T?"

43 • An Old Proverb

"There is an old proverb," said Bravura, "that says that if a cardinal is present, then you can't have a quirky bird without a quacky bird, or a quacky bird without a quirky bird. And if there isn't such a proverb, then there should be! Can you see why the proverb is true?"

44 • A Question

"Is a quacky bird derivable from Q_1 and T?"

45 • An Interesting Fact About the Queer Bird Q

"You have seen that the queer bird Q is derivable from the bluebird B and the thrush T. It is of interest that you can alternatively derive a bluebird B from a queer bird Q and a thrush T. Can you see how? The method is a bit tricky!"

• 46 •

"One can derive a cardinal C from Q and T more easily than from B and T—in fact, you need an expression of only four letters. Can you find it?"

47 • Goldfinches

"Another bird derivable from B and T which I have found useful is the *goldfinch* G defined by the following condition:

$$Gxyzw = xw(yz)$$

"Can you see how to derive it from B and T?"

. . .

"We could go on endlessly deriving birds from B and T," said Bravura, "but it is now getting chilly and Mrs. Bravura has prepared a nice dinner for us. Tomorrow I will tell you about some other birds."

SOLUTIONS

1 · Given a bluebird B, we are to show that for any birds C and D, there is a bird E that composes C with D. Well, BCD is such a bird E, because for any bird x, (BCD)x = ((BC)D)x = C(Dx). Therefore BCD composes C with D.

2 · We saw in the solution to Problem 1 of Chapter 9 that if y is any bird that composes x with M, then x is fond of the bird yy. Now, BxM composes x with M (according to the last problem), and so x must be fond of (BxM)(BxM).

Let us double-check: (BxM)(BxM) = BxM(BxM) = x(M(BxM)) = x((BxM)(BxM))—because M(BxM) = (BxM(BxM)). So (BxM)(BxM) = x((BxM)(BxM)), or what is the same thing, x((BxM)(BxM)) = (BxM)(BxM), which means that x is fond of the bird (BxM)(BxM).

The expression (BxM)(BxM) can be shortened to M(BxM). So x is fond of M(BxM).

3 · We have just seen that for any bird x, x is fond of M(BxM). If we take x to be the mockingbird M, then M is fond of M(BMM). Now, in the solution to Problem 2 in Chapter 9, we saw that any bird of which the mockingbird is fond must be egocentric. Therefore M(BMM) is egocentric.

Let us double-check: M(BMM) = (BMM)(BMM) = BMM(BMM) = M(M(BMM)) = (M(BMM))(M(BMM)). And so we see that M(BMM) = (M(BMM))(M(BMM)), or what is the same thing, (M(BMM))(M(BMM)) = M(BMM), which means that M(BMM) is egocentric.

4 • Since for any bird x, x is fond of M(BxM), then the kestrel K is fond of M(BKM). Therefore M(BKM) is hopelessly egocentric, according to the solution of Problem 9 of Chapter 9, in which we saw that any bird of which the kestrel is fond must be hopelessly egocentric.

5 • It is sometimes easiest to work these problems backward. We are looking for a bird D such that Dxyzw = (xy)(zw). Let us look at the expression (xy)(zw) and see how we can get back to Dxyzw, where D is the bird to be found. Well, we look at the expression (xy) as a unit—call it A—and so (xy)(zw) = A(zw), which we recognize as BAzw, which is B(xy)zw. So the first step of the "backward" argument is to recognize (xy)(zw) as B(xy)zw. Next, we look at the front end B(xy) of the expression and recognize it as BBxy. And so B(xy)zw is BBxyzw. Therefore we take D to be the bird BB.

Let us double-check by running the argument forward.

Dxyzw = BBxyzw, since D = BB.
= B(xy)zw, since BBxy = B(xy).
= (xy)(zw) = xy(zw)

6 • Since we have already found the dove D from B, we are free to use it. In other words, in any solution for B_1 in terms of B and D, we can replace D by BB, thus getting a solution in terms of B alone.

Again we will work the problem backward.

x(yzw) = x((yz)w) = Bx(yz)w. We recognize Bx(yz) as DBxyz, and so Bx(yz)w = DBxyzw. Therefore x(yzw) = DBxyzw, or what is the same thing, DBxyzw = x(yzw). We can therefore take B_1 to be the bird DB. The reader can check the solution by running the argument forward.

In terms of B alone, B_1 = (BB)B, which also can be written B_1 = BBB.

7 • We will use the bird B_1 found in the last problem. Again we will work the problem backward.

xy(zwv) = (xy)(zwv). Looking at (xy) as a unit, we can see that (xy)(zwv) = B_1(xy)zwv. Also B_1(xy) = BB_1xy, so B_1(xy)zwv = BB_1xyzwv. And so we take E to be the bird BB_1.

In terms of B alone, E = BB_1 = B(BBB).

To illustrate a point, suppose we tried to find E directly from B, without using any birds previously derived from B. We could proceed as follows:

We look at the expression xy(zwv). The first thing we try to do is to free the last letter v from parentheses. Well, xy(zwv) = (xy)((zw)v) = B(xy)(zw)v. Now we have freed v from parentheses. We next work on the expression B(xy)(zw), and we would like to free w from parentheses. Looking at B(xy) as a unit, we see that B(xy)(zw) = B(B(xy))zw. We have now freed w from parentheses, and as good fortune would have it we have freed z as well. It remains merely to work on B(B(xy)). We wish to free y from parentheses, but since it is enclosed in two pairs of parentheses, we first free it from the outer pair. Well, B(B(xy)) = BBB(xy). We now look at BBB as a unit and see that BBB(xy) = B(BBB)xy. And so we take E to be B(BBB), which is the same solution we got before.

In this analysis, we have substantially duplicated the labor of deriving the bird B_1, and had this problem been posed *before* Problem 6, we would have had to do this. The moral is that in solving these problems, the reader should be on the lookout for solutions to earlier problems that might be helpful.

8 • Starting from scratch, the solution would be long. Using the eagle of the last problem, the solution is easy:

x(yzwv) = x((yzw)v) = Bx(yzw)v

But Bx(yzw) is EBxyzw, so Bx(yzw)v = EBxyzwv. So we take B_2 to be EB.

In terms of B alone, B_2 = B(BBB)B.

9 · There are two ways we can go about this which will be interesting to compare.

Our first method uses the dove D. Now, xyz(wv) = (xy)z(wv). Looking at (xy) as a unit, we see that (xy)z(wv) = D(xy)zwv. Also D(xy) = BDxy, and so D(xy)zwv = BDxyzwv. And so we take D_1 to be BD, which in terms of B alone is B(BB).

We can also look at the matter this way: xyz(wv) = (xyz)(wv). Looking at (xyz) as a unit, we see that (xyz)(wv) = B(xyz)wv. However, B(xyz) we recognize as B_1Bxyz. Therefore B_1B is also a solution.

Now, B_1 = BBB, so B_1B = BBBB. But BBBB = B(BB), and so we really get the same solution.

10 · We use the bird D_1 of the last problem. Looking at (zw) as a unit, x(y(zw)) = Bxy(zw) = D_1Bxyzw. So we take B_3 to be D_1B.

In terms of B alone, B_3 = B(BB)B.

11 · Again, we can go about this two ways. On the one hand, if we look at (yz) as a unit, then x(yz)(wv) = Dx(yz)wv. Also Dx(yz) = DDxyz, and so we can take D_2 to be DD, which in terms of B is BB(BB).

On the other hand, we can look at x(yz) as a unit and see that x(yz)(wv) = B(x(yz))wv. But B(x(yz)) = B_3Bxyz, and so B_3B is also a solution.

It is really the same solution, since B_3B = B(BB)BB = BDBB = D(BB) = DD, which in turn is BB(BB).

We might remark that we have proved a stronger result than was called for: We were required to derive D_2 from B, but we have in fact succeeded in deriving it from D, since D_2

= DD. Therefore if we were not told that the forest contains a bluebird, but were given only the weaker condition that it contains a dove, this would still be enough to imply that the forest contains a dovekie.

12 · We will prove the stronger result that if the forest contains an eagle (without necessarily containing a bluebird) then it must contain a bald eagle.

Looking at $(y_1y_2y_3)$ as a unit, we see that $x(y_1y_2y_3)(z_1z_2z_3) = Ex(y_1y_2y_3)z_1z_2z_3$. But $Ex(y_1y_2y_3) = EExy_1y_2y_3$, and so $x(y_1y_2y_3)(z_1z_2z_3) = Ex(y_1y_2y_3)z_1z_2z_3 = EExy_1y_2y_3z_1z_2z_3$. And so we take $\hat{E}$ to be EE.

In terms of B, the bird EE is B(BBB)(B(BBB)).

13, 14, and 15 · First, we shall do Problem 14: Given W and I, the bird WI is a mockingbird, because for any bird x, WIx = Ixx = xx, since Ix = x.

Now for Problem 15: Given W and K, the bird WK is an identity bird, because for any bird x, WKx = Kxx = x.

Putting these two problems together, WK is an identity bird, and hence W(WK) should be a mockingbird by Problem 14. Let us check:

W(WK)x = WKxx = (WKx)x = (Kxx)x = xx.

Yes, W(WK) is a mockingbird. This solves Problem 13.

16 · For any bird A whatsoever, the bird CKA is an identity bird, because for any bird x, CKAx = KxA = x. So, for example, CKK is an identity bird; so is CKC.

17 · CI is a thrush, because for any birds x and y, CIxy = Iyx = yx.

18 · The given condition of the problem implies that the thrush T is fond of some bird A. Thus TA = A. Then for any bird x, TAx = Ax. Also TAx = xA, since T is a thrush.

Therefore Ax = xA, and so A commutes with every bird x.

19 · Given the bluebird B and the mockingbird M, as well as the thrush T, we know from Problem 2 of this chapter that T is fond of the bird M(BTM). Remember that for *any* bird x, x is fond of M(BxM). Therefore, according to the last problem, M(BTM) commutes with every bird.

20 · We will work the problem backward: yzx = Tx(yz). We recall the dove D and we see that Tx(yz) = DTxyz. Therefore we take R to be DT. In terms of B and T alone, R = BBT.

21 · Working the problem backward, with only a robin available, we find the solution virtually forced on us! We want to get xzy back into the position xyz. Well, xzy = Ryxz—what else can we do? Now, Ryx = RxRy—again, what other move could we make? Finally, RxR = RRRx.

Retracing our steps, RRRx = RxR, hence RRRxy = RxRy = Ryx. Since RRRxy = Ryx, then RRRxyz = Ryxz = xzy. Therefore we take our cardinal C to be the bird RRR.

A bonus question: When written in terms of B and T, C = (BBT)(BBT)(BBT). This expression can be shortened by one letter: C = RRR = BBTRR = B(TR)R, since BBTR = B(TR). So C = B(T(BBT))(BBT).

The expression B(T(BBT))(BBT) has only eight letters and is Alonzo Church's expression for a cardinal. Personally, I find it easier to remember the cardinal as RRR.

22 · a. Cx = RRRx = RxR

b. Since Cx = RxR and R = BBT, then Cx = BBTxR = B(Tx)R.

23 · Yes; CC is a robin, because CCxy = Cyx, hence CCxyz = Cyxz = yzx.

24 • We will work the problem backwards: zyx = Rxzy = (Rx)zy = C(Rx)yz = BCRxyz. And so we take F to be BCR.

25 • We can also analyze the situation this way: zyx = Tx(zy) = Tx(Tyz) = ETxTyz = (ETx)Tyz = TT(ETx)yz, because (ETx)T = TT(ETx). Continuing, TT(ETx) = ETTETx, hence TT(ETx)yz = ETTETxyz. Therefore we can take F to be ETTET.

26 • If we take F to be BCR, as in Problem 24, then in terms of B and T, the bird F = B(B(T(BBT))(BBT))(BBT).

We get a shorter solution if we express F as ETTET and then reduce to B and T. This is done as follows: ETTET = B(BBB)TTET, because E = B(BBB). Now B(BBB)TTET = BBB(TT)ET = B(B(TT))ET = B(TT)(ET) = B(TT)(B(BBB)T). And so we get a solution shorter by four letters.

27 • zxy = Fyxz = CFxyz, because Fyx = CFxy. We therefore can take V to be CF.

28 • According to law (a) stated in Problem 22, CF = RFR, and CF is a vireo. So RFR is a vireo.

29 • Yes; CV is a finch, because CVxyz = Vyxz = zyx.

30 • For any bird A, the bird RAK must be an identity bird, because RAKx = KxA = x. So, for example, RRK and RKK are both identity birds.

31 • xywz = (xy)wz = C(xy)zw = BCxyzw. And so we take C* to be BC.

32 • Actually, we can get the bird R* from just C*: xzwy = C*xzyw. Also C*xzy = C*C*xyz, therefore C*xzyw = C*C*xyzw. So, xzwy = C*C*xyzw. We therefore take R* to be C*C*.

33 • We can get F* from B, C*, and R* as follows: xwzy = R*xywz = (R*x)ywz = C*(R*x)yzw = BC*R*xyzw, since C*(R*x) = BC*R*x, so we take F* to be BC*R*.

34 • Just as we got V from C and F (V = CF), we can get V* from C* and F*.

xwyz = F*xzyw = C*F*xyzw, because F*xzy = C*F*xyz. And so we take V* to be C*F*.

35 • The secret here is remarkably simple! Take C** to be BC*; R** to be BR*; F** to be BF*; and V** to be BV*.

36 • C*T is a vireo, because C*Txyz = Txzy = zxy. This means that BCT is a vireo.

37 • We can get Q from a bluebird B and a cardinal C as follows:

y(xz) = Byxz = CBxyz, since Byx = CBxy.

And so we take Q to be CB.

In terms of B and T, Q = CB = RRRB = RBR = BBBTBR = B(TB)R = B(TB)(BBT).

38 • We will now find a good use for the starred birds of "some relatives of bluebirds and thrushes." x(zy) = Bxzy = C*Bxyz. We can therefore take Q_1 to be C*B. In terms of B and C, we take Q_1 to be BCB.

39 • y(zx) = Byzx = R*Bxyz. We can therefore take Q_2 to be R*B. In terms of B and C, we take Q_2 to be BC(BC)B or, more simply, C(BCB).

40 • Suppose the forest contains a cardinal C. If a quixotic bird Q_1 is present, a CQ_1 must be a quizzical bird, because CQ_1xyz = Q_1yxz = y(zx). On the other hand, if a quizzical bird Q_2 is present, then CQ_2 must be a quixotic bird, because CQ_2xyz = Q_2yxz = x(zy).

41 • z(xy) = Bzxy = V*Bxyz. We can therefore take Q_3 to be V*B.

However, Q_3 can be gotten directly from B and T much more simply: z(xy) = T(xy)z = BTxyz. And so it is simpler to take Q_3 to be BT.

42 • z(yx) = Bzyx = F*Bxyz. And so we can take Q_4 to be F*B. Another solution follows from the next problem.

43 • Suppose a cardinal C is present. If a quirky bird Q_3 is present, then CQ_3 must be a quacky bird, because CQ_3xyz = Q_3yxz = z(yx). On the other hand, if a quacky bird Q_4 is present, then CQ_4 must be a quirky bird, because CQ_4xyz = Q_4yxz = z(xy).

Since BT is a quirky bird, then C(BT) is a quacky bird, and so for Q_4 we can take C(BT) instead of F*B.

44 • Yes; Q_1T is a quacky bird, since Q_1Txyz = T(yx)z = z(yx).

Since we can take Q_1 to be BCB, then Q_1T = BCBT = C(BT), and we get the same solution as if we took Q_4 to be CQ_3.

45 • QT(QQ) is a bluebird because QT(QQ)xyz = QQ(Tx)yz = Tx(Qy)z = Qyxz = x(yz).

46 • QQ(QT) is a cardinal, since QQ(QT)xyz = QT(Qx)yz = Qx(Ty)z = Ty(xz) = (xz)y = xzy.

47 • xw(yz) = Cx(yz)w = B(Cx)yzw = BBCxyzw. And so we take G to be BBC.

The bird G has some curious properties, as we will see later on.

◐ 12 ◑

Mockingbirds, Warblers, and Starlings

MORE ON MOCKINGBIRDS

Inspector Craig returned early the next morning and again found Professor Bravura in the garden. The first thing that struck Craig was the singing of a distant bird whose song was the strangest that Craig had ever heard. It seemed totally disjointed; first there was a simple melodic line and then, out of the blue, a trill that seemed totally unrelated to the melody. Then followed a melody in a completely unrelated key!

"You've never heard a mockingbird before?" asked Bravura, who noticed Craig's astonishment.

"I guess not! It sounds almost mad!"

"Oh, well," said Bravura, "it remembers bits and snatches from the other birds and doesn't always put them together in the most logical order. I must say, though, that this particular mockingbird sounds wilder than any I've ever heard.

"Let me tell you some combinatorial properties of the mockingbird M," continued Bravura. "It has what is called a *duplicative* effect—it causes repetition of variables. It has this in common with the lark and the warbler. No bird derivable from B and T can have a duplicative effect, so the mockingbird is quite independent of them—it is definitely *not* derivable from B and T. But from the *three* birds B, T, and M, a whole variety of important birds can be derived."

1 • The Bird M_2

"A very simple, but useful, example is the bird M_2—which I sometimes call a 'double' mockingbird—defined by the condition:

$$M_2xy = xy(xy)$$

"This bird is derivable from just B and M. That's pretty obvious, isn't it?"

2 • Larks

"You recall the lark L satisfying the condition $Lxy = x(yy)$. Well, L is derivable from B, T, and M. One way is to derive it from B, C, and M, or from B, R, and M. Can you see how?"

• 3 •

"I might mention, incidentally, that L is also derivable from the bluebird B and the warbler W. Can you see how? Actually, this fact is rather important."

• 4 •

"My favorite construction of a lark," said Bravura, "uses just the mockingbird M and the queer bird Q. It is also the simplest! Can you see how it's done?"

WARBLERS

Just then a warbler flew by.

"Tell me," said Craig, "can a warbler be derived from B, T, and M? Since a lark can, I would not be too surprised if a warbler can."

"Ah, that's a good question," replied Bravura, "and it has a fascinating history. The logician Alonzo Church was interested in the entire class of birds derivable from the four birds B, T, M, and I. My forest happens to follow the thinking of Church; all my birds are derivable from B, T, M, and I. Now, in 1941, Church showed how to derive a warbler from B, T, M, and I. His method was both bizarre and ingenious; his expression for W in terms of B, T, M, and I involved twenty-four letters and thirteen pairs of parentheses! I will tell you about it another time." *Note to reader:* I discuss this in some of the exercises of this chapter.

"Shortly after," continued Bravura, "the logician J. Barkley Rosser found a much shorter expression—one with only ten letters. In looking at his expression, I noticed that he didn't use the identity bird I at all, hence your guess was correct: A warbler can be derived from just B, T, and M. It can be derived even more simply from B, C, and M—and more simply still from B, C, R, and M. But first let me tell you about another bird closely related to W."

5 • The Bird W′

"Show that from B, T, and M you can derive a bird W′ satisfying the following condition:

$$W'xy = yxx$$

"We might call W′ a *converse* warbler," said Bravura. "Curiously enough, W′ is easier to derive than W. It is particularly simple to derive W′ from B, R, and M. Can you see how?"

6 • The Warbler

"Now that you have W′, it is simple to get W. In fact, W can be derived from B, R, C, and M using an expression of only four letters. Can you see how?"

• 7 •

"Now express W in terms of B, T, and M. This can be done with an expression of only ten letters, and there are two such expressions."

8 • A Question

"You now see that W is derivable from B, T, and M. Is a mockingbird M derivable from B, T, and a warbler W?"

9 • Two Relatives of W

"We will occasionally have use for a bird W* satisfying the condition W*xyz = xyzz. How do you derive W* from B, T, and M? And what about a bird W** satisfying the condition W**xyzw = xyzww?"

10 • Warblers and Hummingbirds

"Another bird for which I have found use is the *hummingbird* H defined by the following condition:

Hxyz = xyzy

"Show that H is derivable from B, C, and W—and hence from B, M, and T."

11 • Hummingbirds and Warblers

"You can also derive a warbler from B, C, and H—in fact, you can do it from C and H, and even more simply from R and H. Can you see how?"

STARLINGS

"I have been in this forest some time now," said Craig, "and I have never seen a kestrel. Are there any kestrels here?"

"Absolutely not!" cried Bravura, in an unexpectedly fierce tone. "Kestrels are *not allowed* in this forest!!"

Craig was quite surprised at the severity of Bravura's response and was on the verge of asking him *why* kestrels were not allowed, but he decided that the question might be tactless.

"Ah, there goes a starling," Bravura said more brightly. "Tell me, are you planning to visit the Master Forest?"

"I was planning to visit Curry's Forest," replied Craig.

"And so you should!" replied Bravura. "But you shouldn't stop there; you should continue on until you reach the Master Forest. You will pass through several other interesting forests along the way—before you leave, I'll draw you a map. You will find your experience in the Master Forest to be a true education!"

"Then I'll definitely go," said Craig.

"Good!" replied Bravura. "But I should prepare you for your visit by telling you about the starling, since this bird plays a feature role in the Master Forest."

12 • Starlings

"A starling," said Bravura, "is a bird S satisfying the following condition:

$$Sxyz = xz(yz)."$$

"Why is that bird so important?" asked Craig.

"You will find that out when you reach the Master Forest," replied Bravura.

"Anyway," he continued, "you should know that a starling can be derived from B, T, and M—and more easily, from B, C, and W. The standard expression for S in terms of B, C, and W has seven letters, but I have discovered another having only six letters. It will be helpful to you to use the goldfinch G, which satisfies the condition $Gxyzw = xw(yz)$. The starling S is easily derivable from B, W, and G."

How is this done?

THE STARLING IN ACTION

"You have now seen that S is derivable from B, C, and W," said Bravura. "It is also possible to derive W from B, C, and

S. In fact, W is derivable from just C and S, or alternatively from R and S. I will also show you that W is derivable from T and S."

13 • Hummingbirds Revisited

"You recall that the hummingbird H is defined by the condition Hxyz = xyzy. You have seen that H is derivable from B, C, and W. We now need to find out if a hummingbird is alternatively derivable from S and C—and even more simply from S and R. Is it?"

• 14 •

"Now write down an expression for a warbler in terms of S and R and one in terms of S and C."

"You now see," said Bravura, "that the class of birds derivable from B, C, and S is the same as the class of birds derivable from B, C, and W, since S is derivable from B, C, and W and W is derivable from C and S."

• 15 •

"Since W is derivable from S and C, and C is derivable from B and T, then of course W is derivable from B, T, and S. However, W is derivable from just T and S. Can you show this?"

• 16 •

"Prove that M is derivable from T and S."

"And now," said Bravura, "you see that the class of birds derivable from B, T, and W is the same as the class of birds derivable from B, T, and S, since S is derivable from B, T, and W—it is even derivable from B, C, and W, and in the other direction, W is derivable from T and S. This class of birds is also the same as the class derivable from B, M, and T, since W is derivable from B, M, and T, and in the other direction, M is derivable from W and T, as you have seen.

"More important," said Bravura, "is the fact that the class of birds derivable from B, T, M, and I is the same as the class of birds derivable from B, C, W, and I, since W is derivable from B, T, and M, and in the other direction, T is derivable from C and I—you recall that CI is a thrush. Either of these groups of four birds forms a *basis* for my forest, in the sense that every bird here is derivable from either of the foursomes. Alonzo Church preferred to take B, T, M, I as a basis; Curry preferred the basis B, C, W, I. Alternatively, we could use B, C, S, I as a basis, and for certain purposes this is technically convenient, but you will learn more about that when you reach the Master Forest."

"I am starting out tomorrow," said Craig, "and I am ever so grateful for all you have taught me. It should stand me in good stead in the journey ahead."

"It certainly should," said Bravura. "You have been a diligent student, and it has been a great pleasure to tell you some of the facts about birds I have learned. There are many more birds derivable from B, T, M, and I that I am sure would interest you. I think I will give you these derivations as exercises to take along with you to work out at your leisure. You will also encounter many other such birds in your travels ahead.

"Since you are making your journey on foot, it should take you about three days to reach Curry's Forest. This forest is named after Haskell Curry, and appropriately so, since Curry was both an eminent combinatorial logician and an avid birdwatcher. After Curry's Forest, you will come to Russell's Forest—named for Bertrand Russell. Then you will come to another forest—let's see now, I can never remember its name! Anyhow, next you will arrive at an extremely interesting forest named for Kurt Gödel. These four forests form a chain known as the Forests of Singing Birds. From Gödel's Forest it should take you two days to reach the Master Forest. I wish you the best of luck!"

Here are some of the exercises that Bravura gave to Craig. Sketches of the solutions are given at the end of the chapter.

Exercise 1 (modeled on Church's derivation of W):

a. From B and T, derive a bird G_1 satisfying the condition $G_1xyzwv = xyv(zw)$.

b. From G_1 and M, derive a bird G_2 satisfying the condition $G_2xyzw = xw(xw)(yz)$.

c. From B, T, and I, derive a bird I_2 such that for any bird x, $I_2x = xII$.

d. Show that for any bird x, $I_2(Fx) = x$, where F is a finch.

e. Now show that $G_2F(QI_2)$ is a warbler. *Note:* Q is the queer bird.

Exercise 2 (the standard starling): The standard expression for a starling in terms of B, C, and W is B(B(BW)C)(BB). Show that this really is a starling.

Exercise 3: A *phoenix* is a bird Φ satisfying the condition $\Phi xyzw = x(yw)(zw)$. The bird Φ is standard in combinatory logic. Show that Φ can be derived from S and B. This is tricky! An expression of only four letters works.

Exercise 4: A psi bird is a bird Ψ satisfying the condition $\Psi xyzw = x(yz)(yw)$. The bird Ψ is also standard in combinatory logic. Show that Ψ is derivable from B, C, and W. *Hint:* Let H* be the bird BH. The bird Ψ is easily derivable from H* and the dovekie D_2; remember that $D_2xyzwv = x(yz)(wv)$.

Exercise 5: It is a curious fact that Ψ is derivable from B, Φ, and—of all birds!—the kestrel K. We will divide this problem into two parts:

a. Show that from Φ and B we can get a bird Γ satisfying the condition $\Gamma xyzwv = y(zw)(xywv)$.

b. Show that Ψ is derivable from Γ and K.

Exercise 6: a. Show that from S and one bird already derived from B and T we can get a bird S′ satisfying the condition S′xyz = yz(xz).

b. Show that a warbler is derivable from S′ and the identity bird I.

Exercise 7: There is a bird $\hat{Q}$ derivable from Q alone such that C$\hat{Q}$W is a starling. Can you find it? The expression for it has six letters.

SOLUTIONS

1 · xy(xy) = M(xy) = BMxy, and so we take M_2 to be BM.

2 · x(yy) = x(My) = BxMy = CBMxy, and so CBM is a lark. Also, BxMy = RMBxy, and so RMB is also a lark.

We know that BBT is a robin R, and so BBTMB is a lark. Also BBTM = B(TM), and so B(TM)B is a lark. This gives a fairly simple expression for L in terms of B, T, and M.

3 · x(yy) = Bxyy = W(Bx)y = BWBxy. Therefore BWB is a lark.

4 · x(yy) = x(My) = QMxy, and so QM is a lark!

5 · M_2R is a converse warbler, because M_2Rxy = Rx(Rx)y = Rxyx = yxx.

In terms of B, M, and R, we can take W′ to be BMR. In terms of B, M, T, we can take W′ to be BM(BBT).

We could also take W′ to be B(BMB)T, as the reader can verify.

6 · CW′ is a warbler, because CW′xy = W′yx = xyy. In terms of B, M, C, R, we can take W to be C(BMR). This is Bravura's expression for a warbler.

7 • We showed in Problem 22 of the last chapter that for any bird x, Cx = B(Tx)R. Therefore B(TW′)R is a warbler. If we take BM(BBT) for W′ and BBT for R, we get the expression B(T(BM(BBT)))(BBT); this expression is Bravura's. We could alternatively take B(BMB)T for W′, thus getting B(T(B(BMB)T))(BBT); this is Rosser's expression for a warbler.

8 • Yes; M can even be derived from W and T, because WTx = Txx = xx, and so WT is a mockingbird.

We now see that the class of birds derivable from B, T, and M is the same as the class of birds derivable from B, T, and W.

9 • Take W* to be BW and W** to be B(BW).

10 • xyzy = C*xyyz = W*C*xyz. We therefore take H to be W*C*. In terms of B, C, and W, H = BW(BC).

11 • From H and R we first derive the bird W′. Well, yxx = Rxyx = HRxy. Therefore HR is a converse warbler. Hence C(HR)—or alternatively R(HR)R—is a warbler.

12 • W**G is a starling because W**Gxyz = Gxyzz = xz(yz). So we take W**G for S, which in terms of B, C, and W is the expression B(BW)(BBC).

13 • Yes, it is. SR is a hummingbird, since SRxy = Ry(xy), hence SRxyz = Ry(xy)z = xyzy.

14 • Since SR is a hummingbird, then R(SRR)R is a warbler according to Problem 11. Also C(SRR) is a warbler, and so is C(S(CC)(CC)).

15 • This is particularly simple: ST is a warbler, since STxy = Ty(xy) = xyy.

16 • We have just seen that ST is a warbler. Also, for any warbler W, the bird WT is a mockingbird, as we saw in Problem 8. Therefore STT is a mockingbird.

SOLUTIONS TO THE EXERCISES

Ex. 1: a. Take $G_1 = BG$.
b. Take $G_2 = G_1(BM)$.
c. Take $I_2 = B(TI)(TI)$.
We leave the last two to the reader.
Ex. 2: Left to the reader.
Ex. 3: Take $\Phi = B(BS)B$.
Ex. 4: Take $\Psi = H^*D_2$.
Ex. 5: a. Take $\Gamma = \Phi(\Phi(\Phi B))B$.
b. $\Gamma(KK)$ is a psi bird.
Ex. 6: a. Take $S' = CS$.
b. $S'I$ is a warbler.
Ex. 7: Take $\hat{Q} = Q(QQ(QQ))Q$.

◐ 13 ◑
A Gallery of Sage Birds

While Inspector Craig is wending his way to Curry's Forest, we will take time out to look at a medley of sage birds. But first I must tell you about *combinatorial* birds in general.

By a bird of *order 1* is meant a bird A such that for any bird x, the bird Ax can be expressed in terms of x alone. For example, the mockingbird M is of order 1, since Mx = xx and the expression xx no longer involves the letter M; it is an expression in just the letter x. Another example is the identity bird I, since Ix = x. The birds M and I are the only birds of order 1 that we have so far encountered. Of course, we could construct from the birds of the last chapter an infinite variety of birds of order 1—for example, we might wish to consider a bird A such that Ax = x(xx). The bird WL would work. Or we could construct a bird A such that Ax = (x(xx))((xxx)x)—such a bird would also be of order 1.

By a bird of *order 2* is meant a bird A such that Axy can be expressed in terms of just x and y. Examples are the thrush T, the lark L, and the warbler W; these three birds are obviously of order 2.

A bird of *order 3* is a bird A whose definition involves three variables—say, x, y, z. Thus Axyz is expressible in terms of just x, y, and z. Most of the birds we have so far encountered are of order 3—the birds B, C, R, F, and V and the queer bird Q and its relatives Q_1, Q_2, Q_3, Q_4 are all examples of birds of order 3.

We similarly define birds of order 4, 5, 6, 7, 8, and so forth. Doves are of order 4; the bald eagle Ê is of order 7.

A bird having some order or other is called a *proper combinatorial* bird—or more briefly, a *proper* bird. By a *combinatorial* bird is meant any bird expressible in terms of proper birds. Not every combinatorial bird is proper. For example, the birds T and I are both proper; hence TI is a combinatorial bird, but it is not proper, for if it were, what order could it be? It isn't of order 1, because TIx can be reduced to xI, but no further reduction is possible. TIxy can be expressed as xIy, but we haven't got rid of I, so TI is not of order 2. The best we can do with TIxyz is to express it as xIyz, but the x is still in the way, so no further reduction is possible. No matter how many variables we tack onto the right of TIxyz, we can never get rid of I, so TI is not of any order; hence it is not a proper bird. On the other hand, IT is proper, since IT = T.

SOME SAGE BIRDS

We recall that by a *sage bird* is meant a bird Θ such that for any bird x, if one calls out x to Θ, then Θ will respond by naming a bird of which x is fond—in other words, x(Θx) = Θx (x is fond of Θx).

Sage birds are *not* proper birds! However, sage birds *can* be expressed in terms of proper birds; this can be done in a variety of ways that are quite fascinating. In Chapter 10 we never actually *constructed* a sage bird; we merely proved that if that forest obeyed certain conditions, then a sage bird must *exist* there. We shall now see how to *find* sage birds, given that certain proper birds are present.

• 1 •

Derive a sage bird from a mockingbird M, a bluebird B, and a robin R. This can be done using an expression of only five letters.

• 2 •

Find a five-letter expression for a sage bird in terms of B, C, and M.

• 3 •

A simpler construction of a sage bird uses a mockingbird, a bluebird, and a lark. Can you find it?

• 4 •

Derive a sage bird from a mockingbird, a bluebird, and a warbler.

• 5 •

A tougher job is to derive a sage bird from a bluebird, a cardinal, and a warbler. Care to try it? There are several ways in which this can be done, which will become apparent in the course of this chapter.

ENTER THE QUEER BIRD

We recall that the queer bird Q satisfies the condition Qxyz = y(xz). Thus Qxy composes y with x. Also Qxyz = Byxz. The queer bird is very useful in connection with sage birds.

• 6 •

Show that a sage bird is derivable from a queer bird, a lark, and a warbler.

• 7 •

Now can you see a way to solve Problem 5?

8 • Queer Birds and Mockingbirds

A particularly neat construction of a sage bird uses just the queer bird Q and the mockingbird M. Can you find it?

Discussion: By a *regular* combinator is meant a proper combinator such that, in its definition, the leftmost variable—say, x—of the left side of the equality is also the leftmost variable of the right side and occurs only once on the right side. For example, the cardinal is regular; Cxyz = xzy, and x is the leftmost variable of the right-hand side—xzy—and occurs only once in the expression xzy. On the other hand, the robin R is not regular; Rxyz = yzx, and x is not the leftmost variable of yzx. Also M is irregular, because x occurs twice in xx. The combinators B, C, W, L, S, I, and K are all regular; the combinators T, R, F, V, and Q are all irregular.

In each of the problems 1, 2, 3, 4, we derived a sage bird from three proper combinators; one was irregular, the mockingbird, and the other two were regular. In Problem 7 we derived a sage from three regular combinators. In Problem 8 we derived a sage from two irregular combinators, M and Q. We will now see that a sage can be derived from just two *regular* combinators—moreover, in such a fashion that each of them is derivable from B, C, and W.

CURRY'S SAGE BIRD

9 • Starlings and Larks

Show that a sage bird can be derived from a starling S and a lark L.

10 • Curry's Sage Bird

Now show that a sage bird can be derived from a bluebird, a warbler, and a starling. This can be done using an expression of only five letters.

Note: The solution of the above problem provides a second solution to Problem 5, since S can be derived from B, C, and W.

THE TURING BIRD

A bird deserving particular attention is the *Turing bird* U, defined by the following condition:

$$Uxy = y(xxy)$$

This bird was discovered by the logician Alan Turing in the year 1937, and is one of the most remarkable birds in existence! The reader will soon see why.

11 • Finding a Turing Bird

Before I tell you why I am such an admirer of the Turing bird, let's see if you can *find* one, given the birds B, M, and T, and any birds derivable from them. Can you find a Turing bird?

12 • Turing Birds and Sage Birds

The remarkable thing about the Turing bird U is that from U alone you can derive a sage bird—moreover, you can do it in as simple and direct a manner as can be imagined. Can you see how?

Some open problems: We now see that a sage bird can be derived from just *one* proper combinator—Turing's bird U. Of course, U is not regular. Can a sage be derived from just one *regular* combinator? I tend to doubt it, but I cannot prove that the answer is negative. Can a sage be derived from B and one other regular combinator? This is another question I have not been able to answer. As far as I know, these two problems are open, though I haven't checked the literature sufficiently to be sure of this.

OWLS

13 • Owls

An extremely interesting bird is the *owl* O defined by the following condition:

$$Oxy = y(xy)$$

Show that an owl can be derived from B, C, and W—in fact, from just Q and W.

• 14 •

A sage bird can be derived from O and L. Better yet, a Turing bird is derivable from O and L. How?

• 15 •

Show that a mockingbird is derivable from O and I.

• 16 •

Show that O is derivable from S and I.

WHY OWLS ARE SO INTERESTING

17 • A Preliminary Problem

Preparatory to the next problem, prove that if a bird x is fond of a bird y, then x is fond of xy.

• 18 •

An interesting thing about owls is this: If you call out a sage bird to an owl, the owl will always respond by naming a sage bird—either the same sage bird or a different one. In other words, for any sage bird Θ, the bird $O\Theta$ is also a sage bird. Prove this.

• 19 •

Another interesting thing about owls is that if you call out an owl to a sage bird, the sage bird will respond by naming a sage bird. In other words, for any sage bird Θ and any owl O, ΘO is a sage bird. Prove this.

• 20 •

Equally if not more interesting is the fact that an owl is fond *only* of sage birds! In other words, for any bird A, if $OA = A$, then A must be a sage bird. Prove this.

• 21 •

The last problem has as a corollary a fact that generalizes the result of Problem 19. Let us say that a bird A is *choosy* if it is fond only of sage birds. All owls are choosy, according to the last problem, but there may be other choosy birds. Now let Θ be a sage bird. Prove that it is not only the case that ΘO is a sage, as in Problem 19, but that for *any* choosy bird A, the bird ΘA must be a sage.

22 • Similarity

A bird A_1 is said to be *similar* to a bird A_2 if A_1 and A_2 respond the same way to any bird x—in other words, for every bird x, $A_1x = A_2x$. As far as their responses to birds are concerned, *similar* birds behave identically.

We proved in Problem 18 that for any sage bird Θ, the bird OΘ is also a sage, but we didn't prove that OΘ is necessarily the same bird as Θ. However, OΘ can be proved to be *similar* to Θ. How?

Remarks: A bird forest is called *extensional* if no two distinct birds are similar—in other words, if for any birds A_1 and A_2, if A_1 is similar to A_2, then $A_1 = A_2$. Extensional forests might also be called *sparse*, since it easily follows from the extensional condition that there cannot be more than one identity bird, one mockingbird, one cardinal, one starling, and so forth.

Although the subject of extensionality is an important one, we will not be treating it in this volume. There is one fact, though, that I believe will interest you: In an *extensional* forest, an owl is fond of *all* sage birds! Do you see how to prove this?

I hope you see the ramifications of this! This fact, together with Problem 20, implies that an owl is fond of sage birds and no other birds. Thus, if you go over to an owl O and call out the name of a bird x, if O responds by calling back x, then x is a sage bird; if O calls back some bird other than x, then x is not a sage bird. So, in an extensional forest, owls seem to somehow know which birds are sage birds and which ones are not. Is this not wise of them?

• 23 •

Prove that in an extensional forest, an owl is fond of all sage birds.

SOLUTIONS

1 • Our starting point is that any bird x is fond of the bird M(BxM), as we proved in the solution of Problem 2 of Chapter 11. And so our present problem reduces to finding a bird Θ such that for any bird x, Θx = M(BxM).

Well, BxM = RMBx (R is the robin), so M(BxM) = M(RMBx) = BM(RMB)x. And so we can take Θ to be BM(RMB).

Let us double-check that BM(RMB) really is a sage bird: For any bird x, BM(RMB)x = M(RMBx) = RMBx(RMBx) = BxM(RMBx) = x(M(RMBx)). Since M(RMBx) = BM(RMB)x, then x(M(RMBx)) = x(BM(RMB)x). Therefore BM(RMB)x = x(BM(RMB)x)—they are both equal to BxM(RMBx)—and so BM(RMB) is a sage.

2 • BxM = RMBx, but also BxM = CBMx, and so M(BxM) = M(CBMx) = BM(CBM)x. Since x is fond of M(BxM) and M(BxM) = BM(CBM)x, then x is fond of BM(BCM)x, and so BM(CBM) is also a sage bird.

3 • We proved in Problem 25 of Chapter 9 that x is fond of Lx(Lx), where x is any bird. Now, Lx(Lx) = M(Lx) = BMLx. Hence x is fond of BMLx, which makes BML a sage bird!

Incidentally, this provides an alternative proof for the results of the last two problems:

For *any* lark L, the bird BML is a sage. Now, CBM is a lark, according to Problem 2 of the last chapter, hence BM(CBM) is a sage, which again solves Problem 2. Also RMB is a lark, according to Problem 2 of the last chapter, and so BM(RMB) is a sage, which again solves Problem 1.

4 • Since BWB is also a lark, according to Problem 3 of the last chapter, then by the above problem, BM(BWB) is a sage bird.

5 • We will defer the solution till after the next problem.

6 • Again we use the important fact that x is fond of Lx(Lx). Now, Lx(Lx) = QL(Lx)x. Also, QL(Lx) = QL(QL)x, hence QL(Lx)x = QL(QL)xx, and so Lx(Lx) = QL(QL)xx. Furthermore, QL(QL)xx = W(QL(QL))x. This proves that

Lx(Lx) = W(QL(QL))x, and since x is fond of Lx(Lx), then x is fond of W(QL(QL))x, which means that W(QL(QL)) is a sage bird.

7 · If in the above expression we take BC for Q, we get W(CBL(CBL)), which can be shortened to W(B(CBL)L). We can then take BWB for L, thus getting the expression W(B(CB(BWB))(BWB)).

Another solution will result from a later problem.

8 · Again we use the fact that x is fond of Lx(Lx), and therefore x is fond of M(Lx). Now, M(Lx) = QLMx, so x is fond of QLMx, which means that QLM is a sage bird.

We can now take QM for L, because QM is a lark, as we showed in Problem 4, Chapter 12. We thus get the expression Q(QM)M. And so Q(QM)M is a sage bird, as the reader can verify directly.

9 · It is also the case that Lx(Lx) = SLLx, and so SLL is a sage bird.

10 · We just showed that SLL is a sage. Also SLL = WSL and so WSL is a sage. Since BWB is a lark, we can take BWB for L, thus getting WS(BWB).

This is Curry's expression for a fixed point combinator.

Note: We know that B(BW)(BBC) is a starling, from Problem 12, Chapter 12, and so we can take this expression for S in WS(BWB), thus getting W(B(BW)(BBC))(BWB). This is another expression for a sage in terms of B, C, W, and so we have another solution to Problem 5.

11 · There are many ways of going about this. Here is one. Since the forest contains B, T, and M, it also contains W, L, and Q. Now, y(xxy) = Q(xx)yy = LQxyy = W(LQx)y = BW(LQ)xy. We can therefore take U to be BW(LQ).

12 · For all x and y, Uxy = y(xxy), or what is the same thing, for all y and x, Uyx = x(yyx). We take U for y and we see that UUx = x(UUx). Therefore UU is a sage bird.

13 · y(xy) = Byxy = CBxyy = W(CBx)y = BW(CB)xy. We can therefore take O to be BW(CB).

Also, y(xy) = Qxyy = W(Qx)y = QQWxy, and so QQW is also an owl.

14 · LO is a Turing bird, since LOxy = O(xx)y = y(xxy). And so also LO(LO) is a sage bird.

15 · OIx = x(Ix) = xx, so OI is a mockingbird.

16 · SIxy = Iy(xy) = y(xy), so SI is an owl.

17 · Suppose x is fond of y. Then xy = y. Since x is fond of y and y = xy, then x is fond of xy.

18 · Suppose Θ is a sage bird; we are to show that OΘ is a sage bird.

Take any bird x. Then x is fond of Θx, since Θ is a sage. Therefore, by the last problem, x is fond of x(Θx). But x(Θx) = OΘx, and so x is fond of OΘx. Therefore OΘ is a sage bird.

19 · Suppose Θ is a sage. Then for any bird y, Θy = y(Θy), so in particular, ΘO = O(ΘO). Then for any bird x, ΘOx = O(ΘO)x = x(ΘOx). So ΘOx = x(ΘOx), or equivalently, x(ΘOx) = ΘOx, which means that x is fond of ΘOx. Therefore ΘO is a sage.

20 · Suppose OA = A. Then A = OA, hence for any bird x, Ax = OAx = x(Ax). Since Ax = x(Ax), x is fond of Ax, and so A is a sage.

21 · Suppose A is choosy and Θ is a sage. Since Θ is a sage, then A is fond of ΘA. But since A is fond only of sages, then ΘA must be a sage.

22 · Suppose Θ is a sage. Then for every bird x, Θx = x(Θx). Also OΘx = x(Θx). Therefore OΘx = Θx, since both are equal to x(Θx). Therefore OΘ is similar to Θ.

23 · Suppose the forest is extensional. Now suppose Θ is a sage. By the last problem, OΘ is similar to Θ, and since the forest is extensional, then OΘ *is* the bird Θ. Thus OΘ = Θ, which means that O is fond of Θ. And so in an extensional forest, O is fond of *all* sage birds.