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CANTOR ON THE NOTION OF INFINITY IN SPINOZA AND LEIBNIZ

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Abstract. *In his analysis of the history of philosophy and mathematics, Cantor indicates the early modern period as the one in which, because of the general rejection of the notion of an infinite number, some major prejudices against his new ideas on infinity were developed. However, Spinoza and Leibniz are regarded also as the first authors possessing a correct notion of absolute infinity and actual infinity. After showing how Cantor came to this paradox, in this paper I argue that his views cannot be considered completely accurate: a precise superposition between Cantor's notion of transfinite number and the early moderns' infinite number is impossible, because there is a discrepancy in the notion of set that has not been taken into account. At the same time, Spinoza and Leibniz are under a certain aspect Cantor's precursors, because Cantor adopts Spinoza's terminology to distinguish between absolute infinity and the transfinite, and because Cantor's studies on Leibniz's calculus presented him an example of infinite wholes used in mathematics as an aid for new discoveries.*
Keywords. *Georg Cantor, Spinoza, Leibniz, mathématics, infinity.*

Résumé. Dans son analyse de l'histoire de la philosophie et des mathématiques, Cantor indique le début de la période moderne comme celle dans laquelle, en raison du rejet général de la notion d'un nombre infini, certains préjugés majeurs contre ses nouvelles idées sur l'infini ont été développés. Cependant, Spinoza et Leibniz sont également considérés comme les premiers auteurs possédant une notion correcte de l'infini absolu et de l'infini véritable. Après avoir montré comment Cantor en est arrivé à ce paradoxe, je soutiens que ses vues ne peuvent pas être considérées comme complètement exactes : une superposition précise entre la notion de nombre transfini de Cantor et le nombre infini des premiers modernes est impossible car il y a une divergence dans la notion d'ensemble qui n'a pas été prise en compte. En même temps, Spinoza et Leibniz sont les précurseurs de Cantor parce que Cantor adopte la terminologie de Spinoza pour distinguer l'infini absolu et le transfini et parce que les études de Cantor sur le calcul de Leibniz lui ont fourni un exemple des ensembles infinis utilisés en mathématiques et onsitué ainsi une aide pour de nouvelles découvertes.
Mots-clés. Georg Cantor, Spinoza, Leibniz, mathématiques, infini.

In many passages of his works, Georg Cantor argues that the reflections on the concept of infinity belonging to early modern authors such as Spinoza and Leibniz represented a major obstacle for the development of his own notion before his time. This harsh criticism however contradicts other passages in which Cantor praises Spinoza's and Leibniz's intuitions regarding infinity, arguing that they were the only authors capable of grasping some key aspects of his distinction between absolute infinity and the transfinite. In this paper, I will show the origin of this apparent contradiction and I will explore the relationship between Cantor and these two authors, in order to determine whether Cantor claims were accurate with respect to a more rigorous approach to the history of mathematics and philosophy. Considering their reception, this analysis will also show the nature of Cantor's relationship with Spinoza and Leibniz, determining if it should be qualified as a simple confrontation or as an actual influence.

Transfinite numbers and infinite numbers

The main reason why Spinoza and Leibniz cannot be regarded, in Cantor's mind, as true anticipators of his notion of infinity, is their rejection of the concept of an infinite number. Reflecting on this topic was quite common in early modern times[1] : starting from Galileo Galilei, mathematicians and philosophers debated on whether or not it was possible to conceive a number capable of expressing infinity and, if that was the case, which properties such number would have. Galilei concluded that this concept would inevitably lead to a contradiction, because admitting the possibility of an infinity expressed in numerical form means also admitting that infinity is somehow homogeneous to finite numbers, sharing then with them properties like « being greater than », « being equal to » and « being less than » something else. Intuitively, such properties do not seem to apply very well to the notion of infinity, because for instance the series of natural numbers and the series of their squares, if analysed in the same way of finite sets, don't seem to contain as many elements in the same interval, while at the same time they are both supposed to represent infinite sets, meaning that in infinity they should instead be equal. More precisely, it seems that in the same interval a one-to-one correspondence between the two sets cannot be established[2]. Similar reflections lead Galilei to a position

1. Concerning specifically Leibniz and the infinite number, I've already explored this topic in another article published in this periodical (see M. Brancato, « The young Leibniz and the principle of contradiction : adoption and use between philosophy and mathematics », *Principia rationis. Les principes de la raison chez Leibniz (1646-1716)*, A. Lalanne (dir.) *Lumières*, n°29, 2017).
2. On the topic see E. Knobloch, « Galileo and Leibniz : Different Approaches to Infinity », *Archive for History of Exact Sciences*, Springer, Vol. 54, No. 2 (June 1999), p.87-99, or the

that will be very influential in the early modern period: pertaining infinity, qualifications such as less or greater have no place, meaning that an infinite number homogeneous to its finite numerical counterparts would represent a contradictory concept.

Both Spinoza and Leibniz, in their own terms, rejected the notion of an infinite number in a similar way. Spinoza does so in an important letter to Meyer on infinity, dated 20 April 1663, where the mathematicians are mentioned as guarantors of his opinion :

> Neither Number, nor Measure, nor Time, inasmuch as they are only aids for the imagination, can be infinite. [...] For they have not only discovered many things which cannot be expressed by any number, which shows sufficiently the inadequacy of numbers to determine everything, but they also have many things which cannot be equated with any number, but exceed any number that can be given[3].

Being an author more acquainted with mathematics than Spinoza, there are many more passages in Leibniz's production in which he is openly against infinite numbers. The contradiction attributed to this concept is always linked to the axiom by which the whole is greater than its part, a principle very dear to Leibniz, to the point that he often uses it as the foundation principle for his mathematics :

> Since in this infinite number the even numbers would be as much as the even and the odd numbers taken together, that is as much as the regular numbers, then the infinite number negates that axioms which says that the whole is greater than its part [...] but negating that axiom is impossible, or, which is the same, that axiom never, or never if not in nothingness fails, so the infinite number is impossible[4].

As it is widely known instead, the novelty of Cantor's approach rests precisely on the possibility of comparing infinite sets without being dissuaded by the apparent contradiction highlighted by Leibniz. Cantor introduced three different notions of infinity: an infinity pertaining God only, called absolute infinity, an infinity *in concreto*, i.e. the transfinite,

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more recent S. Levey, « Comparability of Infinities and Infinite Multitude in Galileo and Leibniz », *G.W. Leibniz, Interrelations between Mathematics and Philosophy*, N. Goethe, P. Beeley, D. Rabouin (dir.), Archimedes, vol 41. Dordrecht, Springer, 2015.

3. B. Spinoza, *The Correspondence of Spinoza*, London, Allen & Unwin LTD, 1928, p.119-120 (Letter XXIX in the classic Latin edition B. Spinoza, *Opera posthuma*, Amsterdam, 1677, p.468-469).
4. G. W. Leibniz, *Accessio ad arithmeticam infinitorum*, A II, 1, p.348-349. Where otherwise indicated, the English translations are provided by the author of this paper.

which pertains to infinitely many actual things, and infinity *in abstracto*, i.e. the transfinite numbers that allow us to deal with the actual infinity of the transfinite. It is clear that in this context the notion of a transfinite number is essential because it establishes differences between infinite sets, which is the main idea that sets Cantor apart from his predecessors. Having always been particularly interested in the story of the notion of infinity, it is not surprising then that Cantor associates his own notion of transfinite numbers to the historical-mathematical notion of infinite number: they both assume the possibility of infinite wholes, concepts that refer to the infinite sets they represent, without losing their distinction with other similar sets, i.e. without following Galilei's assumptions on infinity that deny any possible comparison. As a result, Cantor has a precise idea of Spinoza's and Leibniz's roles in the history of this notion : « This is the origin of the wrong opinion, spread in almost the entirety of early modern philosophy by Descartes, Spinoza, and Leibniz, according to which there are no infinite numbers, or better, there are no transfinite numbers »[5].

Cantor's statement cannot be taken lightly since he also connects the mistakes of the early modern philosophers to the general aversion of contemporary mathematicians to his innovative ideas[6]. It follows that any possible positive influence of Spinoza and Leibniz over Cantor's ideas has to be found in other aspects of his theory, not related to infinite numbers or transfinite numbers, or in passages from Spinoza and Leibniz's works that openly contradict their opinions on infinite numbers.

Spinoza's absolute infinity

Cantor's relationship with Spinoza could be easily seen as an actual influence, instead of a simple confrontation. The main reason is that references to Spinoza are present in Cantor's texts since the very beginning of his career, specifically in his abilitation on number theory awarded at the University of Halle in 1869. Here, presented as one of the final *theses*, we read : « Iure Spinoza mathesi (Eth. pars. I. prop. XXXVI, app.) eam vim tribuit, ut hominibus norma et regula veri in

5. G. Cantor, letter to Mittag-Leffler, G. Cantor, *Briefe*, Berlin, Springer-Verlag, 1980, p.211.
6. *Ibid.*

omnibus rebus indagandi sit »[7]. Later in Cantor's production, as it will be shown, Spinoza's importance is mainly associated with his concept of an absolute infinity pertaining to God, something that Cantor openly accepts :

> I make a distinction between a « Infinitum aeternum sive Absolutum », which refers to God and its attributes, and a "Infinitum creatum sive Transfinitum", which is present every time an infinite created in nature has to be established, as concerning my firm belief that the number of created individuals is actually infinite, both in the universe and on our earth, and most likely in every part of it, about which I completely agree with Leibniz[8].

While Leibniz's role will be discussed in the next chapter, from this quote follows that, if we were to argue for a proper influence of Spinoza over Cantor's theories, the passage of the young Cantor's dissertation has to lead in some way to the concept of absolute infinity and its distinction with the transfinite. The part of Spinoza's *Ethics* quoted by Cantor is the famous ending of part I, in which Spinoza argues against a finalistic approach concerning God's intervention in our world. The fact that the topic of this passage seems to be very far from the number theory thematised in Cantor's abilitation is not necessarily contradictory. Presenting theses completely unrelated to the topic of the dissertation was a common habit in German abilitations, even in early modern times: they were often beliefs that the author wanted to restate on that occasion, with no necessary correlation with the actual topic of his work.

However, the hypothesis of a stronger influence is more likely[9] : in the appendix at the end of *Ethics* part I, there is indeed a reference to the mathematical model as the perfect example for true knowledge, the one stemming from the intellect in opposition to the notions elaborated by the imagination[10]. Far from being a passage simply celebrating science

7. G. Cantor, « De transformatione formarum ternariarum quadraticarum », *Gesammelte Abhandlungen mathematischen und philosophischen Inhalts*, Berlin, Springer-Verlag, 1932, p.62.
8. Cantor to Franzlein, G. Cantor, *Briefe*, ouvr. cit., p.254.
9. In their paper, P. Bussotti and C. Tapp elaborate further the parallel between Spinoza and Cantor, arguing for an even stronger influence based also from other passages of Spinoza's *Ethics* (see P. Bussotti, C. Tapp, « The influence of Spinoza's concept of infinity on Cantor's set theory », *Studies in History and Philosophy of Science*, 40, 2009, p.25-35).
10. « So they took it for certain that the judgments of the Gods far surpass human understanding. And this alone would have caused the truth to be hidden from the human race forever, if mathematics, which is not concerned with purposes but only with the essences and properties of figures, had not disclosed a different criterion of truth to human beings » (B.

through the mathematical model, it is already present here a common reasoning in Spinoza that prevents God from having any determination that would invalidate its perfection. In the case of a God possessing a will through which he would cause the creation of the world, Spinoza argues that : « this doctrine takes away the perfection of God. For if God acts for the sake of a purpose, he is necessarily seeking something that he lacks »[11]. Spinoza concludes the appendix with a famous argument that can surely be interpreted, in Cantor's approach, as a premise to his notion of absolute infinity :

> To those who ask why God did not create all men to be governed by the guidance of reason alone, I simply answer that he had no lack of material for creating all things from the highest degree of perfection to the lowest ; or more properly speaking, because the laws of nature itself were so ample that they sufficed to produce everything that can be conceived by an infinite intellect[12].

The conceivability of as many things as there are, the infinite intellect and the idea that every determination would undermine God's absolute nature can very well be taken as premises to Cantor's further investigations on the notion of infinity because they are all concepts connected with the dichotomy between an undetermined infinity and a determined one.

In his mature writings, Cantor's opinion about Spinoza emerges more clearly, and it is based mainly on Spinoza's letter XXIX to Meyer[13] on the nature of infinity previously quoted. As it is known, Spinoza here establishes a clear distinction between an infinity by its own nature and an infinity conceived as something limitless. The first notion of infinity is associated with the concept of substance and the concept of eternity, while the other notion is associated with modes and the concept of duration. As a consequence, the concepts of greater and smaller are allowed only when dealing with modes, while the absolute nature of the

Spinoza, *Ethics Proved in Geometrical Order*, Cambridge, Cambridge University Press, 2018, p. 37).

11. B. Spinoza, *Ethics*, ouvr. cit., p. 38.
12. B. Spinoza, *Ethics*, ouvr. cit., p. 41.
13. In composing a list of various passages written by early modern philosophers on the notion of infinity, Cantor specifies : « I reserve to myself a detailed discussion of these passages and, especially, of the most important and meaningful letter from Spinoza to L. Meyer for another occasion » (G. Cantor, *Gesammelte Abhandlungen mathematischen und philosophischen Inhalts,* ouvr. cit., p.175). This important letter mentioned by Cantor is indeed letter XXIX.

one and only substance negates this very possibility. At the same time, the concept of number is associated with the faculty of imagination, meaning that it cannot in any way express the absolute infinity and it can express the infinity of the modes only if they are considered under a certain aspect. Integrating this letter with other parts of his *Ethics*, especially proposition 21, Spinoza seems to distinguish between three kinds of infinity : the infinity of the substance and its attributes, the infinity of some modes which follows from « the absolute nature of any of God's attributes »[14], and the infinity seen as an indefinite concept unable to be determined by numbers, thus pertaining the imagination only.

While the last notion of infinity is closely connected to Spinoza's rejection of the infinite number, his different distinctions resemble convincingly Cantor's different accounts of infinity previously mentioned, to the point that in the distinction between absolute and actual infinity Cantor embeds Spinoza's terminology[15] :

> What I claim and believe in this work, as much as in my previous attempts, is that after the finite there exist a *Transfinitum* (that could be defined also as a *Suprafinitum*), that is there exist an unlimited series of determined modes, which by their own nature are not finite but infinite, and, just like the finite, can be determined through certain numbers, well-defined and distinguishable[16].

In Cantor's interpretation then, Spinoza was right in postulating infinite modes, which correspond more or less to Cantor's notion of the transfinite, because they can be manipulated in certain ways without losing their properties, as instead happens to the absolute infinity. At the same time, Spinoza's fault, according to Cantor, was not being able to see that from the first two concepts of infinity follows the existence of infinite numbers and not their rejection, i.e. to the transifinite *in concreto* always corresponds a series of transfinite numbers *in abstracto*.

14. B. Spinoza, *Ethics,* ouvr. cit., p. 23.
15. Against a simple adoption of Spinoza's theories, in another passage Cantor specifies : « A particularly difficult aspect of Spinoza's system is the relationship between finite Modes and infinite Modes. It stays there, unexplainable, how and under which circumstances the finite can affirm itself against the infinite, or the infinite against the finite » (G. Cantor, *Gesammelte Abhandlungen mathematischen und philosophischen Inhalts*, *op. cit.*, p.177). While nonetheless adopting the terminology then, Cantor saw himself as someone perfecting Spinoza's distinctions, rather than a simple adopter.
16. G. Cantor, *Gesammelte Abhandlungen mathematischen und philosophischen Inhalts*, ouvr. cit., p.176.

Spinoza's impact on Cantor's theory then can be indeed conceived as an important influence, at least with regards to the terminology adopted to distinguish the different kinds of infinity, but only if we accept Cantor's interpretation, based on the idea that Spinoza openly contradicts himself when advocating against the infinite number.

Apart from the adoption of the terminology, the difference between the absolute infinity of God and the determined infinity of the modes might appear too familiar to truly set Spinoza apart from other authors : after all, it might be argued that a similar distinction can also be observed in other common philosophical traditions, reinterpreted in the light of Christianity, where it is reimagined as the opposition between an infinite, absolute and sometimes unknowable God and the infinity accessible to finite creatures. However, Cantor rightfully perceives a fundamental difference: because of the influence of the Aristotelian tradition, in these traditions to the infinity accessible to human beings was associated the idea that nature abhors the infinity *in actu*. While surely from a wider historical perspective this position contributed to the general rejection of the infinite number, also shared by Spinoza, his approach is significantly different on this specific topic. From the fact that in Spinoza's account there exist one and only substance and this substance is infinite, follows that the infinity of the attributes and the infinity of the modes considered in their own nature, because they are derived from God's infinity, can very well be conceived as forms of infinity *in actu*, positioning Spinoza against the Scholastic tradition. This interpretation is suggested by the letter to Meyer, while the association between absolute infinity and actual infinity is not distinctly present in Spinoza's *Ethics*[17].

Also, setting aside for a moment Cantor's interpretation, while there seem to exist a distintion between absolute and determined infinity in Spinoza, the idea of an unknowable God does not cerntainly follow from it : qualifying the divine infinity as « indetermined » is certainly a stretch, even if its concept is derived in opposition to the actual determined infinity of the modes, because God remains in Spinoza's approach *causa sui*. Cantor's insistence on the opposition of this two concepts is based in my opinion more on a certain interpretation of Spinoza that has

17. An analysis of this problem is present in Y. Melamed, « "A Substance Consisting of an Infinity of Attributes" : Spinoza on the Infinity of Attributes », *Infinity in Early Modern Philosophy*, O. Nachtomy, R. Winegar (dir.), Springer, 2018.

its roots in Hegel's famous criticism : Spinoza's concept of substance lies above everything else that exists without truly iteracting with it in any way. In the same fashion, Cantor interprets Spinoza's absolute God as an undetermined entity because it lacks any relation with the determined entities constituting the actual infinity. This undetermined divine infinity, seen by Cantor in Spinoza, strongly influenced his later religious insights and his notion of inconsistent multiplicity, especially in its connection with the Burali-Forti paradox. Here, an analysis similar to that of the infinite number takes place[18], but it will not be explored in the present article, because Cantor's reflections on the early moderns were always focused more on the concepts of infinite number and actual infinity, or on their rejection.

The existence of this actual infinity was enough to convince Cantor of the novelty of Spinoza's approach. Having perceived this distinction then, Cantor's attention shifts from the concept of absolute to the actuality of infinity, and this is the main reason why, in his mind, Spinoza and Leibniz are uniquely connected and different from other early modern authors.

Leibniz's fictional use of infinite wholes and the limits of Cantor's interpretation

Cantor's interpretation of Leibniz falls along the lines of Spinoza's : he is praised because he endorses actual infinity, a fundamental requisite of Cantor's transfinite, but at the same time he is criticised because he seems to contradict himself when rejecting the concept of an infinite number.

While, unlike Spinoza, the status of Leibniz's production in Cantor's times was far from being accessible in its entirety, Cantor shows nonetheless a decent knowledge of Leibniz's works, quoting them mainly from the editions published by Erdmann and Pertz[19]. Some works that he found there, like for instance the *Nouveaux Essais sur l'entendement humain*, argue convincingly against the infinite number, just like other less-known texts quoted in this paper. There is a passage Cantor

18. A good introduction to the topic is C. Menzel, « Cantor and the Burali-Forti paradox », *The Monist*, vol. 67, no. 1, Oxford University Press, 1984, p. 2–107.
19. The adoption of these editions is again evident from the list compiled by Cantor in G. Cantor, *Gesammelte Abhandlungen mathematischen und philosophischen Inhalts,* ouvr. cit., p. 175.

particularly liked, quoted both in his works and in the correspondence, that shows clearly his interpretation :

> Even if I quoted many passages of Leibniz's works in §5, where he declares himself against the infinite numbers, saying among other things : « Il n'y a point de nombre infini ni de ligne ou autre quantité infinie, si on les prend pour des Touts véritables ». « L'infini véritable n'est pas une modification, c'est l'absolu ; au contraire, dès qu'on modifie on se borne ou forme un fini » (about which I concur with him in this last passage on the first statement, but I do not agree about the second one), on the other hand, I am in the lucky position of agreeing with this thinker where, in a sense in contradiction with himself, he talks about the real infinity (different from the absolute) in unequivocal terms. So he says in Erdm. page 118 :
>
>> Je suis tellement pour l'infini actuel, qu'au lieu d'admettre que la nature l'abhorre, comme l'on dit vulgairement, je tiens qu'elle l'affecte partout, pour mieux marquer les perfections de son Auteur. Ainsi je crois qu'il n'y a aucune partie de la matière qui ne soit, je ne dis pas divisible, mais actuellement divisée; et par conséquent la moindre particelle doit être considérée comme un monde plein d'une infinité de créatures différentes[20].

Just like in Spinoza's case then, Cantor does not understand how from these metaphysical premises Leibniz is unable to draw adequate conclusions in mathematics[21]. Cantor's opinion on both Spinoza and Leibniz appears now clear :

> In their evaluation of the finite and the infinite, they essentially agree in those passages that the concept of number includes its finitude, and on the other hand in the true infinite or absolute, which is in God, it is not allowed any determination. On this last point I completely agree, since it cannot be otherwise, because the sentence "omnis determinatio est negatio" is out of the question to me; on the other hand, I see in the first point, as I already said about the Aristotelian reasons in favour of the "*infinitum actu*", a *petitio principii*, which explains some of the contradictions that one could

20. *Ibid*., p. 179. The same quote appears in Cantor's letter to Wundt, dated May 10th 1883 : « On the other hand, I claim for my concept of infinite numbers that it is free from any arbitrariness and it results from the abstraction of reality with the same necessity of the regular finite integers, that until now were used exclusively as the origin of the other mathematical concepts. The transfinite integers are not at all, as you say, "fictions" or "logical postulates" […]. In order to understand this, I do not assume any vision of the world other than the one about which Leibniz's words are authoritative : "Je suis tellement pour l'infini actuel […]." » (G. Cantor, *Briefe*, ouvr. cit., p.136)
21. The struggle in understanding Leibniz's position is well represented in Arthur's hypothetical dialogue between the two authors (see R. T. W. Arthur, « Leibniz in Cantor's Paradise », *Leibniz and the Structure of Sciences. Boston Studies in the Philosophy and History of Science*, V. De Risi (dir.), vol 337., Cham, Springer, 2019, p.71-109).

> find in all these authors, especially in Spinoza and Leibniz. The hypothesis that there should not be other modifications with respect to the absolute […] is also in contradiction with some statements made by these last two philosophers[22].

The solution found by Cantor consists of perfecting the contradictory approach of Spinoza and Leibniz because despite their difficulties they came closer to the correct notion of infinity than Kant, or other philosophers that came after them[23]. The analysis of Cantor's interpretation of Leibniz would end then here, if it was not for the fact that, being Leibniz a much better mathematician than Spinoza, his opinions on the infinite number and on similar topics cannot be dismissed as easily. They are the result of many reflections that, precisely because they do not always appear coherent between themselves, might hide in some cases intuitions closer to Cantor's approach than Cantor himself is willing to admit.

While it is true that Leibniz rejects the infinite number as much as Spinoza already in an early writing such as the *Accessio*, this does not prevent him to wonder about the opposite possibility, to the point of arguing that, if a thing such as an infinite number exists, it would be much better conceived as the number 0, rather than the number 1, as Galilei once suggested. The idea is that, instead of conceiving the infinite number as a very big number, maybe it could be better represented by a number that can take part in an infinite amount of operations without altering itself. In this sense, the number 1 as much as the number 0 can for instance be raised to the power of any possible number and still be equal to themselves. Without focusing too much on how Leibniz develops this idea, it is any case true that a rejection that wonders on the opposite possibility, still offering a rigorous approach when analysing it, is already a step forward, from the point of view of the history of mathematics, towards the use of infinite wholes.

In Leibniz's works, similar cases can be found in which a contradictory idea connected to the notion of infinity can still be useful if adopted in a certain way :

> Speaking philosophically, I maintain that there are no more infinitely small magnitudes than there are infinitely large ones, that is, no more

22. G. Cantor, *Gesammelte Abhandlungen mathematischen und philosophischen Inhalts*, ouvr. cit., p. 175-176.
23. *Ibid.*, p. 177.

> infinitesimals than infinituples. For I hold both to be fictions of the mind through an abbreviated way of speaking, adapted to calculation, as imaginary roots in algebra are too. Meanwhile, I have demonstrated that these expressions are very useful for abbreviating thought and thus for discovery, and cannot lead to error, since it suffices to substitute for the infinitely small something as small as one wishes, so that the error is smaller than any given, whence it follows that there can be no error[24].

In Leibniz's account then, infinitesimals behave similarly to their opposites, infinitely large quantities that are close to the notion of infinite numbers. He is then perfectly aware of the usefulness of infinite wholes :

> It would be a mistake to try to suppose an absolute space which is an infinite whole made up of parts. There is no such thing : it is a notion which implies a contradiction; and these infinite wholes, and their opposites the infinitesimals, have no place except in geometrical calculations, just like the imaginary roots in algebra[25].

Leibniz's opinion on such notions might look negative at first glance, yet he concedes to them much more than any other author of his time, so much that his advancements concerning imaginary roots for instance are taken in great consideration by scholars. In this passage, infinite wholes are presented as opposed to the infinitesimals in the sense that the former deal with the notion of infinitely great quantities, while the latter deal with the notion of infinitely small quantities, but they are both contradictory in the same way as if they are sharing the same nature. However, the fact that they have no place except in geometrical calculations means also that they can indeed have a place in geometrical calculations, showing an approach that does not dismiss completely the usefulness of the infinite wholes dear to Cantor but reinterprets them as useful fictions[26]. The term « fiction » might be misleading, but from the point of view of the history of mathematics a useful fiction used in

24. Leibniz to Des Bosses, G. W. Leibniz, GP II, p. 305. The translation is taken from G. W. Leibniz, *The Leibniz-Des Bosses Correspondence*, New Haven and London, Yale University Press, 2007.
25. G. W. Leibniz, *Nouveaux Essais sur l'entendement humain*, A VI, 6, p. 158.
26. In a recent paper, Richard Arthur and David Rabouin explain convincingly how, since Leibniz's early years, infinitesimals and infinite wholes, despite being described as contradictory and non-existent, can, under certain circumstances, be used in a successful way (cf. R. T. W. Arthur, D. Rabouin, « Leibniz's syncategorematic infinitesimals II : their existence, their use and their role in the justification of the differential calculus », *Archive for History of Exact Sciences*, 74 (5), 2020, p. 401-443).

a coherent way as if it was real can be seen nonetheless as a step towards Cantor's theory.

It is then legitimate to wonder if Cantor was aware of these nuances and to try and understand how he interpreted them. After all, Cantor's interest in Leibniz was not only philosophical, but he actively studied his works from a mathematical standpoint to determine the evolution of the infinitesimal calculus. While he correctly understands Leibniz's fictional yet useful use of infinitesimals, it seems however that the fact that Leibniz labels these extraordinary uses as fictions prevented Cantor from appreciating the novelty of using infinite wholes in a period in which they were for the most part completely rejected :

> The reason why the so-called *potential infinite* or *syncategorematic* (Indefinitum) does not originate such distinction is that it has a meaning only as a relation, as an auxiliary representation of our thought, but it does not denote in itself an *Idea*; in this role, it has shown its value as a means of knowledge and tool of our minds in the differential and integral calculus invented by Leibniz and Newton; the same infinite cannot assume any further meaning[27].

Cantor is much more concerned to side with Leibniz in showing the fictional status of infinitesimals because he believes that they do not share the same metaphysical grounding with his transfinite numbers : any proper use of an infinite whole has to be connected with the idea that it derives from an actual, non-fictional notion of infinity. Even if this distinction is valid from the point of view of Cantor's theory, it is perhaps too strict in a wider context where the aim would be evaluating if Leibniz suggested to Cantor possible uses of a non-standard notion of infinity. In this sense, a more rigorous study on Cantor's reception of the infinitesimal calculus would cast a new light on the nature of Leibniz's influence.

Cantor's failure in seeing how, despite the differences, Leibniz comes close to his use of infinity under certain circumstances leads to the possibility that perhaps also the association between contemporary transcendent numbers and modern infinite numbers is not as effective as Cantor thought. In the contemporary notion suggested by Dedekind and endorsed by Cantor, a number is a symbol denoting a specific set of elements. Different sets can be conceptually different, like the set

27. G. Cantor, *Gesammelte Abhandlungen mathematischen und philosophischen Inhalts*, ouvr. cit., p. 373.

of odd numbers and the set of even numbers, but they can denote the same number of elements, even in infinity, because the same cardinality between the two sets is grounded in the possibility of establishing a 1-1 correspondence between their elements, at least ideally through a determined pattern: it is a concept similar to the Fregean equinumerosity.

In both Spinoza and Leibniz's notions instead the concept of number is not based on these assumptions, which however are required to deal adequately with Cantor's concept of infinity. In the account of these authors, there is always something more to infinity that numbers cannot show properly and, when they try to do so, this inevitably leads to inconsistencies on the notion of set, where a subset happens to have the same cardinality of the set from which it is derived, as in the case of natural numbers and their squares or natural numbers and even or odd numbers. As it is widely known, in Cantor's notion of infinity, the equality between an infinite set and its proper subset is instead the defining property of an infinite set : it is the very reason why, even in infinity, we are allowed to conceive different sets. The whole theory is built around the concept of a 1-1 correspondence as a verifying tool. In this way, Cantor discovers for example that the set of natural numbers and the set of their squares have the same cardinality, while the same cannot be shown between natural numbers and real numbers, leading to the idea that the latter are infinite in a different order.

As much as Leibniz's works on the foundations of mathematics can be easily considered the most advanced that the early modern period had to offer, the concept of a 1-1 correspondence is far from being the main focus, if we understand this concept in a very rigorous way. It is true for instance that in Leibniz's reflections on Galilei there is something about comparing different sets of numbers that can be seen as the prelude to the idea of a 1-1 correspondence, but its lack of rigour leads to contradictions in infinity. For example, the closest idea to a 1-1 correspondence proposed by Leibniz is hidden in his concepts of homogeneity and similarity, which are used sometimes to define geometrical objects capable of estimating different quantities by means of superposition and comparison: it is the case of Leibniz's definition of number. In the act of superposing a line segment to another line segment in order to estimate their lengths, in a sort of « point-to-point correspondence », there is indeed something intuitive that resembles the 1-1 correspondence method, which is already a great result for that time considering the rigorous way in which Leibniz tries to depict the process. However, these similarities are completely lost in infinity, where for example the superposition of two straight lines, one representing the infinite progression of natural numbers and one representing the infinite progression of real numbers, would not possibly show intuitively a discrepancy in the correspondence that would suggest a different order of infinity. The idea of order or position seems to prevail here over the idea of a 1-1 correspondence. However, only possessing a clear notion of bijection, it would be possible to redefine the concept of equality in a way in which the equality between two infinite sets does not invalidate Leibniz's principle by which the whole is greater than its part[28]. In some of Leibniz's texts the tension between the novelty of some notions surfacing into his mind and the limits of a concept of set not based on bijection is evident, as in a 1679 letter to Weigel :

> I believe $\frac{1}{0}$ is infinitely less than the sum of the series 1 1 1 1 1 1 1 1 etc. I think then that between that ordinary infinite, which consists in the collection of every unity, and the finite, that is unity, something intermediate is given, that is $\frac{1}{0}$[29].

With $\frac{1}{0}$ Leibniz means the series $\frac{1}{2}\,\frac{1}{3}\,\frac{1}{4}\,\frac{1}{5}\,\frac{1}{6}\,\frac{1}{7}$ etc., which wouldn't denote in Cantor's theory an infinite set different from that of natural numbers, because the 1-1 correspondence is still possible, but despite these limits, Leibniz's observation was ahead of his time, as it is demonstrated in contrast with Weigel's response, which again mentions Galilei's idea that in infinity such distinctions are not given, censoring in this way his former pupil.

It might appear pointless to remind that Leibniz could not have possessed the notions of a mathematician living in Cantor's times, but it is from this very premise that Cantor's association between his transfinite numbers and the early modern's infinite numbers begin to fall short. When Leibniz states for instance that « numbers do not in themselves go absolutely to infinity, since then there would be a greatest number »[30], he is not thinking about that greatest number as something which entails a conceptually defined set. In fact, he is not considering in this case numbers like ω, ℵ, or even constants or unknown variables,

28. On this topic see M. van Atten, « A note on Leibniz' argument against infinite wholes », *British Journal for the History of Philosophy*, 19 (1), 2011, p. 121-129.
29. G. W. Leibniz, A II, 1, N. 212 and N. 215.
30. G. W. Leibniz, A VI, 3, p. 503.

so much so that, when asked to imagine a hypothetical infinite number, Leibniz feels ha has to pick between a very big « regular » number, the number 1 or the number 0. Even in the example present in the letter to Weigel, as much as we would like to say that $\frac{1}{0}$ is an anticipation of Cantor's concept, $\frac{1}{0}$ is nothing more than a symbol representing a series of rational numbers. The oddity that Leibniz perceives is not based on their 1-1 correspndences but on an idea of order that seems violated by the progression. These were the options available during the time the young Leibniz writes and any further development is connected to this unique approach, combining philosophy and mathematics.

There is a significant difference then between the problem transfinite numbers want to solve and the problem highlighted by the early modern notion of an infinite number. The latter, in the form addressed by the majority of authors, can be summarised as follows: is it possible to conceive among natural or rational numbers a natural or rational number which can effectively express infinity ? To this question, even Cantor would answer negatively, despite his adoption of infinite wholes. Many scholars have debated about the similarities and differences between Cantor, Spinoza, and Leibniz, but I believe that the improper superposition of infinite numbers and transfinite numbers has not been adequately thematised. I concede that the two concepts are not too distant, in fact I concede that from the wider point of view of the history of mathematics and philosophy having turned a problem easily dismissed by the many in a deeper reflection on infinite wholes and their use in calculations, despite their fictional or actual state, is already an important achievement of Leibniz's genius. At the same time, arguing (as Cantor did) that Leibniz or Spinoza did not understand correctly the concept of an infinite number because this concept is supposed to represent the early modern version of the concept of a transfinite number is in my opinion a mistake based on the conscious or unconscious belief that a certain notion of set and cardinality was already available to Spinoza or Leibniz at that time.

Conclusion

Cantor's interpretation of Spinoza and Leibniz rests on two main premises: the first one is that the concept of an infinite number is the early modern counterpart to the contemporary concept of transfinite number and the second one is that from the existence of an actual infinity derives the existence of actual infinite wholes. Both premises prevented Cantor from having an adequate historical perspective on both philosophers. The fact that the first premise is assumed by the father of the transfinite does not make it less or more true from a historical standpoint, but it explains the apparent contradictions in Cantor's claims: to him, rejecting the notion of an infinite number is an unforgivable sin that cannot be redeemed even by the adoption of actual infinity. Furthermore, the second premise characterises Spinoza and Leibniz's positions as unintelligible and contradictory.

However, Cantor's debt to these authors goes far beyond his claims: from the adoption of Spinoza's terminology to describe the transfinite and transfinite numbers through modes to the endorsement of Leibniz's notion of an actual infinity. Leibniz was also the first philosopher that showed the usefulness of infinite wholes in the mathematics studied by Cantor, even if their fictional nature was never accepted: despite the differences, experimenting with these notions showed a path that lead to Cantor's infinity.

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